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Generalized Quasi-Variational Inequality and Implicit Complementarity Problems

by
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Abstract

A new problem called the *generalized quasi-variational inequality problem* is introduced. This new formulation extends all kinds of variational inequality problem formulations that have been introduced and enlarges the class of problems that can be approached by the variational inequality problem formulation. Existence results without convexity assumptions are established and topological properties of the solution set are investigated. A new problem called the *generalized implicit complementarity problem* is also introduced which generalizes all the complementarity problem formulations that have been introduced. Applications of generalized quasi-variational inequality and implicit complementarity problems are given.

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1. Introduction

The importance of the theory as well as the applications of the variational inequality and the complementarity problem has been well documented in the literature. In recent years, various extensions of these two problems have been proposed and analyzed. The most general extension combining the variational inequality and complementarity problems is the one by Chan and Pang [5]. They introduced the quasi-variational inequality and implicit complementarity problems inspired by the work of Mosco [33] who considered the dependence of the function domain on the variable and the work of Fang and Peterson [16] who extended the single-valued function under consideration to a point-to-set mapping.

Although the extensions mentioned above are general, they did not include the possible interaction between the function value and the variable. In this respect, Parida and Sen [37] perhaps were the first ones to extend the variational inequality problem to the generalized variational-like inequality problem for multifunctions taking this possible interaction into consideration.

The aim of this report is to introduce a further extension of the classical variational inequality and complementarity problems from a theoretical standpoint. Our generalized problems which will again be called the generalized quasi-variational inequality and generalized implicit complementarity problems respectively include those problems introduced by Chan and Pang [5], Parida and Sen [37], Fang and Peterson [16], Saigal [41], and Karamardian [25] as special cases. It will also be seen that our generalized problems have a broader range of applications.

In Section 2 we review some definitions of continuity of point-to-set mappings. We also review some concepts on convex sets and convex functions. In particular, we note the fact that any compact convex subset of $\mathbf{R}^n$ is an acyclic absolute neighborhood retract. We cite a fixed point theorem due to Eilenberg and Montgomery [15] which plays an important role in establishing existence results for generalized quasi-variational inequality problems. Finally we give some notations that will be used throughout this report.

In Section 3 we first give a short introduction on variational inequality problems. Then in Section 3.1 we introduce the formulation of the generalized quasi-variational inequality problem which is a unification of all types of variational inequality problems in finite-dimensional spaces that have previously been introduced and we obtain some general existence results for this general problems. In Section 3.2 we introduce the general concepts of copositivity and monotonicity of a point-to-set mapping with respect to another point-to-set mapping, and obtain some existence results for this general problem under the assumption of coercivity, copositivity or monotonicity of

the point-to-set mappings. Finally, in Section 3.3, we investigate several properties of the solution set of the $GQVIP$.

In Section 4 we introduce the formulation of the generalized implicit complementarity problem and establish a relationship between the generalized quasi-variational inequality and the generalized implicit complementarity problems. As a by-product of the results in Section 3, we obtain some existence results for this general problem.

In Section 5 we consider some possible applications of the generalized quasi-variational inequality and implicit complementarity problems. The major areas of our applications are mathematical programming and equilibrium programming. The applications are: minimization problems involving "invex" functions, generalized dual problems and saddle point problems, equilibrium problems involving markets with utility, equilibrium problems involving abstract economies, generalized Nash equilibrium problems and quasi-variational inequality problems of obstacle type. In all these applications we require relatively weak conditions to ensure the existence of solutions to the problems under consideration.

2. Notations and Preliminaries

In this report, $\mathbf{R}^n$ denotes the n-dimensional Euclidean space with the usual inner product $\langle x, y\rangle$ of $x, y \in \mathbf{R}^n$ and norm $\|x\|$ of $x \in \mathbf{R}^n$. The nonnegative orthant $\mathbf{R}^n_+$ is the subset of $\mathbf{R}^n$ consisting of all vectors with nonnegative components. The set of positive integers will be denoted by $\mathbf{N}$. For $K \subset \mathbf{R}^n$, $\text{int}(K)$ and K^c denote the interior and complement of K, respectively. For $K, B \subset \mathbf{R}^n$, $\text{int}_K(B)$ and $\partial_K(B)$ denote the relative interior and relative boundary of B in K, respectively. For any $x, y \in \mathbf{R}^n$, $x \geq (>)\ y$ if and only if $x_i \geq (>)\ y_i$ for all components of x and y. The field of complex numbers is denoted by $\mathbf{C}$. Upper case letters (e.g., F) denote point-to-set maps and lower case letters (e.g., f) denote single-valued functions.

There are four definitions of continuity for point-to-set maps that have been introduced in the literature. We list two of them that are related to our discussion in this report. Let X and Y be Hausdorff spaces and F a point-to-set map from X into Y.

Definition 2.1 (Berge [3]) *The map F is said to be upper semicontinuous (u.s.c.) at $x \in X$ if and only if for any open neighborhood O of $F(x)$, there is a neighborhood V of x such that $F(u) \subseteq O$ for each $u \in V$.*

Definition 2.2 (Hogan [22]) *The map F is said to be upper continuous (closed) at $x \in X$ if and only if a sequence $\{x_n\}$ converging to x, and a sequence $\{y_n\}$ with $y_n \in F(x_n)$ converging to y, implies $y \in F(x)$.*

The relations of the above two definitions can be seen from the following two lemmas.

Lemma 2.3 (Delahaye and Denel [12]) *Suppose $F(x)$ is closed. If F is upper semicontinuous at x, then F if upper continuous at x.* □

A topological space is said to be *first countable* if it has a countable base.

Lemma 2.4 (Delahaye and Denel [12]) *Suppose Y is first countable and there exists a countable neighborhood base at $x \in X$. Also suppose the closure of $Y \setminus F(x)$ is compact. If F is upper continuous at x, then F is upper semicontinuous at x.* □

Definition 2.5 (Hogan [22]) *The map F is said to be lower continuous (open) at $x \in X$ if and only if for any sequence $\{x_n\}$ converging to $x \in X$ and $y \in F(x)$, there exists an n_0 such that the sequence $\{y_n\}$ converging to $y \in Y$ and $y_n \in F(x_n)$ for all $n \geq n_0$.*

It is clear that if F is upper continuous at x, then $F(x)$ is closed. Indeed, suppose $y_n \in F(x)$ and $y_n \longrightarrow y$. By considering the constant sequence $x_n = x$, it follows immediately that $y \in F(x)$. Hence $F(x)$ is closed. F is said to be upper (lower) continuous if F is upper (lower) continuous at every point $x \in X$ and F is continuous if it is both upper and lower continuous. The map F is said to be *uniformly compact near x* if there exists a neighborhood of x, V such that $F(V) = \bigcup_{u \in V} F(u)$ is bounded. We say F is uniformly compact on X if it is uniformly compact near x for all $x \in X$. The following lemma is a direct consequence of Lemma 2.4 and a result due to Berge [3 , Theorem 3, p.110].

Lemma 2.6 *Suppose $X \subseteq \mathbf{R}^n$ and $Y \subseteq \mathbf{R}^m$. Let $F : X \longrightarrow Y$ be an upper continuous point-to-set map such that F is uniformly compact on X. If D is compact, then $F(D) = \bigcup_{x \in D} F(x)$ is also compact.* □

Remark. If the condition of uniform compactness in Lemma 2.6 is replaced by the condition that F is compact valued, then the result of Lemma 2.6 may fail to hold. To see this, consider the following example. Let $X = [0,1]$. Let F be a point-to-set mapping from X into $\mathbf{R}$ defined by

$$F(x) = \begin{cases} \{0\} & \text{if } x = 0 \\ \{1/x\} & \text{if } 0 < x \leq 1 \end{cases}$$

Then F is upper continuous and $F(x)$ is compact for all $x \in X$. Clearly X is compact whereas $F(X)$ is unbounded. Note that F is not *u.s.c.*, and if F is considered as a single-valued function, then it is also not continuous.

A *topological pair* (X, A) consists of a topological space X and a subspace $A \subseteq X$. A map $f : (X, A) \longrightarrow (Y, B)$ between topological pairs is a continuous function from X to Y such that $f(A) \subseteq B$. Given a topological pair (X, A), we let $(X, A) \times I$ denote the pair $(X \times I, A \times I)$ where $I = [0,1]$. Let $X' \subseteq X$ and suppose that $f_0, f_1 : (X, A) \longrightarrow (Y, B)$ agree on X' (that is, $f_0|X' = f_1|X'$). Then f_0 is *homotopic* to f_1 relative to X', denoted by $f_0 \simeq f_1$ rel X', if there exists a map $g : (X, A) \times I \longrightarrow (Y, B)$ such that $g(x, 0) = f_0(x)$ and $g(x, 1) = f_1(x), \forall\ x \in X$ and $g(x, t) = f_0(x), \forall\ (x, t) \in X' \times I$. If $X' = \emptyset$, we omit the phrase "relative to $\emptyset$". The following examples are from Spanier [42, p.23, 24].

Example 2.7 Let $X = Y = \mathbf{E}^2 = \{z \in \mathbf{C} : |z| \leq 1\}$ and let $A = B = \mathbf{S}^1 = \{z \in \mathbf{C} : |z| = 1\}$. Define $f_0 : (\mathbf{E}^2, \mathbf{S}^1) \longrightarrow (\mathbf{E}^2, \mathbf{S}^1)$ to be the identity map and $f_1 : (\mathbf{E}^2, \mathbf{S}^1) \longrightarrow (\mathbf{E}^2, \mathbf{S}^1)$ to be reflection in the origin. Then $g : f_0 \simeq f_1$ rel 0 through the homotopy g defined by $g(re^{i\theta}, t) = re^{i(\theta + t\pi)}$.

Example 2.8 Let X be an arbitrary space and let Y be a convex subset of $\mathbf{R}^n$. Let $f_0, f_1 : X \longrightarrow Y$ be maps which agree on some $X' \subseteq X$. Then $f_0 \simeq f_1$ rel X', because the map $g : X \times I \longrightarrow Y$ defined by

$$g(x,t) = tf_1(x) + (1-t)f_0(x)$$

is a homotopy relative to X' from f_0 to f_1.

A topological space X is said to be ***contractible*** if the identity map of X is homotopic to some constant map of X to itself. A homotopy from 1_X (the identity map of X) to the constant map of X to $x_0 \in X$ is called a *contraction* of X to x_0. Example 2.8 shows that any convex subset of $\mathbf{R}^n$ is contractible. Any set that is starshaped at some point x_0 is also contractible. If A and B are contractible, then both $A \times B$ and $A \cap B$ are contractible (see e.g. Spanier [42, Corollary 8, p.25]). The idea of a contractible space is that it can be deformed continuously into a one-point space. To see that the class of contractible sets in $\mathbf{R}^n$ contains nonconvex sets, consider the following example (Spanier [42, p.26]). Let

$$Y = \{(x,y) \in \mathbf{R}^2 : \ 0 \leq y \leq 1, \ x = 0, 1/n; \ y = 0, \ 0 \leq x \leq 1, \ n \in \mathbf{N}\}.$$

Let $g : Y \times I \longrightarrow Y$ be defined by $g((x,y),t) = (x,(1-t)y)$. Then g is a homotopy from 1_Y to the projection of Y to the x-axis. Since the latter map is homotopic to a constant map, Y is contractible, whereas Y is not convex.

A subset A of X is called a *retract* of X if the inclusion map $i : A \longrightarrow X$ has a left inverse in the category of topological spaces and continuous maps. Hence A is a retract of X if and only if there is a continuous map $r : X \longrightarrow A$ such that $r(x) = x, \forall\, x \in A$. Such a map r is called a *retraction* of X to A. A space Y is said to be an ***absolute retract*** (or ***absolute neighborhood retract***) if , given a normal space X, closed subset $A \subseteq X$ and a continuous map $f : A \longrightarrow Y$, then f can be extended over X (or f can be extended to some neighborhood of A in X). The following lemma will be useful.

Lemma 2.9 *The product of arbitrarily many absolute retracts (or finitely many absolute neighborhood retracts) is itself an absolute retract (or absolute neighborhood retract).*

Proof. (i) Let A be index set and let Y_α be an absolute retract for each $\alpha \in A$. The Cartesian product of the sets Y_α is the set

$$\prod_{\alpha \in A} Y_\alpha = \{x : A \longrightarrow \bigcup_{\alpha \in A} X_\alpha : \ x(\alpha) \in X_\alpha, \forall\, \alpha \in A\}.$$

We write x_α instead of $x(\alpha)$. Let π_α be the projection map of $\prod_{\alpha\in A} Y_\alpha$ on Y_α. It is well known that $f : Y \longrightarrow \prod_{\alpha\in A} Y_\alpha$ is continuous if and only if $\pi_\alpha \circ f$ is continuous for each $\alpha \in A$ (see e.g. Willard [43, Theorem 8.8]). Now given a normal space X , closed subset $B \subseteq X$, and a continuous map $f : B \longrightarrow \prod_{\alpha\in A} Y_\alpha$, the composite map $\pi_\alpha \circ f : B \longrightarrow Y_\alpha$ is continuous for each $\alpha \in A$. Since Y_α is an absolute retract, there is a continuous map $f_\alpha : X \longrightarrow Y_\alpha$ such that $f_\alpha|B = \pi_\alpha \circ f$, for all $\alpha \in A$. Now define $g : X \longrightarrow \prod_{\alpha\in A} Y_\alpha$ by $(g(x))_\alpha = f_\alpha(x), \forall \alpha \in A$. Since $\pi_\alpha \circ g = \pi_\alpha \circ f$ is continuous for all $\alpha \in A$, g is continuous. Also $(g|B)_\alpha = f_\alpha|B = \pi_\alpha \circ f, \forall\, \alpha \in A$. Hence $g|B = f$. Therefore $\prod_{\alpha\in A} Y_\alpha$ is an absolute retract.

(ii) Let $Y_1, \ldots, Y_n$ be absolute neighborhood retracts. Suppose we are given a normal space X, closed subset $B \subseteq X$, and a continuous map $f : B \longrightarrow \prod_{k=1}^n Y_k$. Then $\pi_k \circ f : B \longrightarrow Y_k$ defined

by $(\pi_k \circ f)(x) = f_k(x)$ $(f(x) = (f_1(x), \ldots, f_n(x)))$ is continuous for $k = 1, \ldots, n$. Since Y_k is an absolute neighborhood retract for each k, there exists a neighborhood A_k of B and a continuous map $g_k : A_k \longrightarrow Y_k$ such that $g_k|B = \pi_k \circ f$. Let $C = \bigcap_{k=1}^n A_k$ and $g : C \longrightarrow \prod_{k=1}^n Y_k$ defined by $g(x) = (g_1(x), \ldots, g_n(x))$. Then C is a neighborhood of B, and g is continuous. It is clear that $g|B = f$. Therefore $\prod_{k=1}^n Y_k$ is an absolute neighborhood retract. □

Corollary 2.10 *For all positive integers n, $\mathbf{R}^n$ is an absolute retract.*

Proof. That fact that $\mathbf{R}$ is an absolute retract follows from Tietze's Extension Theorem (see e.g. Willard [43, 15.8]). The result then follows directly from Lemma 2.9. □

Lemma 2.11 *A retract of an absolute retract (absolute neighborhood retract) is an absolute retract (absolute neighborhood retract).*

Proof. Let Y be an absolute retract and $B \subseteq Y$ be a retract. Suppose that we are given a normal space X , a closed subset A of X, and a continuous map $f : A \longrightarrow B$. Let $i : B \longrightarrow Y$ be the inclusion map. Since B is a retract of Y, there exists a continuous function $r : Y \longrightarrow B$ such that $r \circ i = 1_B$. Then $r \circ i \circ f : A \longrightarrow Y$ is continuous. Since Y is an absolute retract, there exists a continuous map $g : X \longrightarrow Y$ such that $g|A = r \circ i \circ f$. Then $r \circ g : X \longrightarrow B$ is continuous and clearly $(r \circ g)|A = f$. Hence B is an absolute retract. For the case where Y is an absolute neighborhood retract, the proof is the similar. □

Lemma 2.12 *Any closed convex subset of $\mathbf{R}^n$ is a retract.*

Proof. Let S be a closed convex subset of $\mathbf{R}^n$. Define $p : \mathbf{R}^n \longrightarrow S$ by $p(x) = y$ where $||x - y|| = \min_{u \in S} ||x - u||$. Then p is a contraction (see, e.g., [38, p.340]). Consequently, S is a retract of $\mathbf{R}^n$. □

A compact metric space X is said to be *acyclic* if (1) $X \neq \emptyset$, (2) the homology group $H_n(X)$ vanishes for all $n > 0$, and (3) the reduced 0-th homology group $\tilde{H}_0(X)$ vanishes. It is true that any compact contractible space is acyclic but not conversely (see e.g. Spanier [42, p.163]). By Lemmas 2.11, 2.12 and Corollary 2.10 and, we have the following corollary. Note that it is clear that any absolute retract is also an absolute neighborhood retract.

Corollary 2.13 *Any nonempty compact convex subset of $\mathbf{R}^n$ is an acyclic absolute retract and hence an acyclic absolute neighborhood retract.* □

The following theorem by Eilenberg and Montgomery turns out to be very useful in our discussion.

Theorem 2.14 (Eilenberg and Montgomery [15, Theorem 2]) *Let M be an acyclic absolute neighborhood retract, N a compact metric space, $r : N \longrightarrow M$ a continuous single-valued mapping and $T : M \longrightarrow M$ a multi-valued upper continuous mapping such that the sets $T(x)$ are acyclic for all $x \in M$. Then the combined (multi-valued) mapping $r \circ T : M \longrightarrow M$ has a fixed point.* □

If we take $N = M$ and $r(x) = x, \forall\ x \in M$, then we have the following theorem.

Theorem 2.15 *Let M be an acyclic absolute neighborhood retract and $T : M \longrightarrow M$ a multi-valued upper continuous mapping such that all the sets $T(x)$ are acyclic for all $x \in M$. Then T has a fixed point.* □

It is worth noting that Theorem 2.15 implies Kakutani's Fixed Point Theorem, since every nonempty compact convex subset is an acyclic absolute neighborhood retract by Corollary 2.13.

Let $f : S \longrightarrow \mathbf{R}$, where S is a nonempty convex subset in $\mathbf{R}^n$. The function f is said to be *quasiconvex* if , for any $x, y \in S$, the following inequality is true:

$$f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}, \forall\ \lambda \in [0, 1].$$

Clearly every convex function is also quasiconvex but not conversely.

Lemma 2.16 *Let S be a nonempty convex set in $\mathbf{R}^n$ and $f : S \longrightarrow \mathbf{R}$ be quasiconvex. Let $A = \{x \in S : f(x) = \min_{u \in S} f(u)\}$. If A is nonempty, then A is convex.*

Proof. Let $x, y \in A$. We then have

$$\max\{f(x), f(y)\} \leq f(u), \forall\ u \in S.$$

Since f is quasiconvex, for any $\lambda, 0 \leq \lambda \leq 1$ we have

$$f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\} \leq f(u), \forall\ u \in S.$$

So $\lambda x + (1 - \lambda)y \in A$, $\forall\ 0 \leq \lambda \leq 1$. Hence A is convex. □

For a nonempty subset $K \subset \mathbf{R}^n$, the *convex hull* of K, denoted by $\mathrm{Co}(K)$, is a convex set spanned by K. That is,

$$\mathrm{Co}(K) = \{\sum_{i=1}^{n} \lambda_i x_i : \lambda_i \geq 0,\ x_i \in K, \forall\ i, \sum_{i=1}^{n} \lambda_i = 1\}.$$

A nonempty subset K of $\mathbf{R}^n$ is a *cone* if $\lambda x \in K, \forall\ x \in K, \lambda \geq 0$. A cone K is *pointed* if $K \cap (-K) = \{0\}$. Let K be a closed convex cone in $\mathbf{R}^n$. Then $K^* = \{y \in \mathbf{R}^n : \langle y, x\rangle \geq 0, \forall\ x \in K\}$ is called the *polar cone* of K. It is easy to see that if $\mathrm{int}(K) \neq \emptyset$, then K^* is also a solid cone. A set $K \subset \mathbf{R}^n$ is said to be *solid* if it has a nonempty interior with respect to some topology in $\mathbf{R}^n$.

Let K be a convex set in $\mathbf{R}^n$ and $f : K \longrightarrow [-\infty, +\infty]$ be a convex function. A vector x^* is said to be a *subgradient* (see, e.g. Rockafaller [38]) of f at a point x if

$$f(z) \geq f(x) + \langle x^*, z - x\rangle, \forall\ z \in K.$$

The set of all subgradients of f at x is called the *subdifferential* of f at x and is denoted by $\partial f(x)$. When $f(x) = \delta(x|K)$, that is, f is the *indicator function* of K, then $x^* \in \partial\delta(x|K)$ if and only if $x \in K$ and $\langle x^*, z - x\rangle \leq 0$ for all $z \in K$. Thus $\partial\delta(x|K)$ is the *normal cone* to K at x (empty if $x \notin K$).

3. The Generalized Quasi-Variational Inequality Problem

We begin this section by giving a short introduction on variational inequality problems. Given a subset K of $\mathbf{R}^n$ and a function f from $\mathbf{R}^n$ into itself, the *variational inequality problem*, denoted by $VIP(f, K)$ is to find a vector $\bar{x} \in K$ such that

$$\langle x - \bar{x}, f(\bar{x})\rangle \geq 0, \forall\ x \in K.$$

This original problem has been extensively studied in the past years. For example, see Eaves [14], Moré [30], and Pang [36]. Basically, the task of the above problem is to find a vector $\bar{x} \in K$ such that the image of $\bar{x}$ under the function f will form an angle less than or equal to 90° with any vector with tail $\bar{x}$ and head $x \in K$.

The variational inequality problem is found to be important in many applications. For instance, let K be a closed convex subset of $\mathbf{R}^n$ and let f be differentiable on a neighborhood of K. It is well known that f is convex on C if and only if

$$f(x) \geq f(y) + \langle \nabla f(y), x - y\rangle$$

for all x and y in K, where ∇f is the gradient of f. If y solves $VIP(\nabla f, K)$, then from the above *gradient inequality*, we see that y solves the following mathematical programming problem

$$\min_{x \in K} f(x).$$

Therefore, the variational inequality problem encompasses the minimization problem.

The theory of variational inequalities was initially proposed for the study of partial differential equations (see, e.g., Hartman and Stampacchia [20]). Much of this early work concentrated on the study of free boundary value problems, which were usually formulated as variational inequality problems over infinite dimentional spaces.

Given a set K in $\mathbf{R}^n$ and a point-to-set mapping F from $\mathbf{R}^n$ into itself, the *generalized variational inequality problem* introduced by Fang and Peterson [16], denoted by $GVIP(F, K)$ is to find a vector $\bar{x} \in K$ and a vector $\bar{y} \in F(\bar{x})$ such that

$$\langle x - \bar{x}, \bar{y}\rangle \geq 0, \forall\ x \in K.$$

We note that the $GVIP(F, K)$ is a different generalization of the $VIP(f, K)$.

Inspired by the work of Mosco [33] and the work of Fang and Peterson [16], Chan and Pang [5] considered the following generalized variational inequality problem. Given two point-to-set mappings X and F from $\mathbf{R}^n$ into itself, the *generalized quasi-variational inequality problem*, denoted by $GQVIP(X, F)$ is to find a vector $\bar{x}$ and a vector $\bar{y} \in F(\bar{x})$ such that

$$\langle x - \bar{x}, \bar{y}\rangle \geq 0, \forall\ x \in X(\bar{x}).$$

Recently Parida and Sen [37] introduced the following generalized variational-like inequality problem for point to set mapping. Let K and C be subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively. Given two maps $\theta : K \times C \longrightarrow \mathbf{R}^n$ and $\tau : K \times K \longrightarrow \mathbf{R}^n$, and a point-to-set mapping $F : K \longrightarrow C$, the *generalized variational-like inequality problem*, denoted by $GVIP(F, \theta, \tau, K, C)$ is to find $\bar{x} \in K, \bar{y} \in F(\bar{x})$ such that

$$\langle \theta(\bar{x}, \bar{y}), \tau(x, \bar{x})\rangle \geq 0, \forall\ x \in K.$$

Using this problem formulation, Parida and Sen [37] were able to establish some existence results for dual problems and saddle point problems. We note that such progress can not be made by using other problem formulations directly.

3.1. Problem Formulation and Some Existence Theorems

Inspired by the work that has been done in the area of variational inequality problems, it is natural for us to consider the following generalized quasi-variational inequality problem which extends all the above variational inequality, generalized variational inequality, quasi-variational inequality, generalized quasi-variational inequality, and generalized variational-like inequality problems. Given K and C subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively, X a point-to-set mapping from K into itself and F a point-to-set mapping from K into C, θ a single-valued function from $K \times C$ into $\mathbf{R}^n$ and τ a single-valued function from $K \times K$ into $\mathbf{R}^n$, *the generalized quasi-variational inequality problem*, denoted by $GQVIP(X, F, \theta, \tau, K, C)$ is to find $\bar{x} \in X(\bar{x}), \bar{y} \in F(\bar{x})$ such that

$$\langle \theta(\bar{x}, \bar{y}), \tau(x, \bar{x}) \rangle \geq 0, \forall\ x \in X(\bar{x}).$$

If $\tau(x, y) = x - y$, then $GQVIP(X, F, \theta, \tau, K, C)$ reduces to the problem of finding $\bar{x} \in X(\bar{x}), \bar{y} \in F(\bar{x})$ such that

$$\langle \theta(\bar{x}, \bar{y}), x - \bar{x} \rangle \geq 0, \forall\ x \in X(\bar{x}),$$

which we denote by $GQVIP(X, F, \theta, K, C)$.

We note that $GQVIP(X, F, \theta, \tau, K, C)$ reduces to $GVIP(F, \theta, \tau, K, C)$ if $X(x) = K$ for all $x \in K$. The $GQVIP(X, F, \theta, \tau, K, C)$ reduces to $QVIP(X, F)$ if we set $K = C = \mathbf{R}^n, \theta(x, y) = y, \tau(x, y) = x - y$. By letting $K = C = \mathbf{R}^n, \theta(x, y) = y, \tau(x, y) = x - y$ and F a single-valued function, the $GQVIP(X, F, \theta, \tau, K, C)$ reduces to $QVIP(X, f)$. Finally, if we set $X(x) = K$ for all $x \in K, \theta(x, y) = y$ and $\tau(x, y) = x - y$, and F a single-valued function f, then $GQVIP(X, F, \theta, \tau, K, C)$ reduces to $VIP(f, K)$. Therefore, it can be seen that our formulation of the generalized quasi-variational inequality problem extends all kinds of variational inequality problem formulations.

The following is important in establishing existence results for $GQVIP(X, F, \theta, \tau, K, C)$.

Theorem 3.1.1 *Let $K \subset \mathbf{R}^n$ be a compact contractible absolute neighborhood retract and $C \subset \mathbf{R}^m$ be a closed contractible absolute neighborhood retract. Let X be a nonempty-valued continuous point-to-set mapping from K into itself and F a contractible-valued upper continuous and uniformly compact point-to-set mapping from K into C. Let φ be a continuous single-valued function from $K \times C \times K$ into $\mathbf{R}$. suppose that*

(i) *there exists a compact contractible absolute neighborhood retract H such that $F(K) \subset H \subset C$.*

(ii) $\varphi(x, y, x) \geq 0, \forall\ x \in K$.

(iii) *for each fixed $(x, y) \in K \times C$, the set*

$$V(x, y) = \{u \in X(x) : \varphi(x, y, u) = \min_{s \in X(x)} \varphi(x, y, s)\}$$

is contractible.

Then there exist $\bar{x} \in X(\bar{x}), \bar{y} \in F(\bar{x})$ such that

$$\varphi(\bar{x}, \bar{y}, x) \geq 0, \forall\ x \in X(\bar{x}).$$

Proof. It follows from Lemma 2.9 and Corollary 2.13 that $K \times H$ is an acyclic absolute neighborhood retract. Now let G be a point-to-set mapping from $K \times H$ into itself defined by

$$G(x, y) = (V(x, y), F(x)).$$

Then $G(x, y)$ is contractible for all $(x, y) \in K \times H$. We claim that G is upper continuous. Suppose (x_n, y_n) converges to (x, y) and $(v_n, w_n) \in G(x_n, y_n)$ converges to (v, w). Then for each n,

$$\varphi(x_n, y_n, s) \geq \varphi(x_n, y_n, v_n), \forall\ s \in X(x_n). \tag{1}$$

For each z in $X(x)$, since X is continuous, there exist n_0 such that z_n converges to z with $z_n \in X(x_n), \forall\ n \geq n_0$. From (1), we have

$$\varphi(x_n, y_n, z_n) \geq \varphi(x_n, y_n, v_n), \forall\ n \geq n_0.$$

By passing to the limit, we then have

$$\varphi(x, y, z) \geq \varphi(x, y, v).$$

Also it is clear that $v \in X(x)$ and $w \in F(x)$. Therefore $(v, w) \in G(x, y)$ and consequently G is upper continuous. Therefore by Theorem 2.14, there exists $(\bar{x}, \bar{y}) \in G(\bar{x}, \bar{y})$. Hence $\bar{x} \in X(\bar{x}), \bar{y} \in F(\bar{x})$ and

$$\varphi(\bar{x}, \bar{y}, x) \geq \varphi(\bar{x}, \bar{y}, \bar{x}) \geq 0, \forall\ x \in X(\bar{x}). \square$$

Remarks.

(i) If C is compact, then the condition that F is uniformly compact is unnecessary. Because in this case, $H \subset C$ is clearly compact.

(ii) If C is compact, then condition (i) of Theorem 3.1.1 is unnecessary. Because in this case, we merely let $H = C$ in the above proof and proceed with the same argument.

(iii) If the set $\mathrm{Co}(F(K)) \cap C$ is a retract of C, then we can let $H = \mathrm{Co}(F(K)) \cap C$. Since K is compact and F is upper continuous and uniformly compact, H is compact and contractible by Lemma 2.6. Also by Lemma 2.11, H is an absolute neighborhood retract.

(iv) If C is convex, then condition (i) of Theorem 3.1.1 holds automatically. Because in this case, the projection $\mathrm{P}_{\mathrm{Co}(F(K))}(\cdot)$ establishes that $\mathrm{Co}(F(K))$ is a retract of C.

(v) If F is *not uniformly compact*, then the conclusion of Theorem 3.1.1 may fail to hold. For example, let $K = [1, 2], C = \mathbf{R}$. Let X be the constant point-to-set mapping K and let F be defined as

$$F(x) = \begin{cases} \{1/(x-1)\} & \text{if } 1 < x \leq 2 \\ \{-1\} & \text{if } x = 1 \end{cases}$$

Finally, let $\varphi(x, y, u) = \langle y, u - x \rangle$. Then all conditions in Theorem 3.1.1 except that F is uniformly compact are satisfied. But it is easy to see that there is no $\bar{x} \in K$ such that $\varphi(x, F(\bar{x}), x) \geq 0,\ \forall\, x \in K$.

We now have the first existence result for the $GQVIP(X, F, \theta, \tau, K, C)$.

Theorem 3.1.2 *Let $K \subset \mathbf{R}^n$ be a compact contractible absolute neighborhood retract and let $C \subset \mathbf{R}^m$ be a closed contractible absolute neighborhood retract. Let X be a nonempty-valued continuous point-to-set mapping from K into itself and F a contractible-valued upper continuous and uniformly compact point-to-set mapping from K into C. Let $\theta : K \times C \longrightarrow \mathbf{R}^n$ and $\tau : K \times K \longrightarrow \mathbf{R}^n$ be continuous single-valued functions. Suppose that*

(i) *there exists a compact contractible absolute neighborhood retract H such that $F(K) \subset H \subset C$,*

(ii) $\langle \theta(x, y), \tau(x, x) \rangle \geq 0, \forall\, (x, y) \in K \times C$,

(iii) *for each fixed $(x, y) \in K \times C$, the set*

$$V(x, y) = \{u \in X(x) : \langle \theta(x, y), \tau(u, x) \rangle = \min_{s \in X(x)} \langle \theta(x, y), \tau(s, x) \rangle\}$$

is contractible.

Then there exists a solution to the $GQVIP(X, F, \theta, \tau, K, C)$.

Proof. By letting $\varphi(x, y, u) = \langle \theta(x, y), \tau(u, x) \rangle$, the result follows directly from Theorem 3.1.1. □

Remarks.

(i) Condition (ii) will be satisfied if, for example, $\tau(x, x) = 0\ \forall\, x \in K$.

(ii) Condition (iii) will be satisfied if, for example, $\langle \theta(x, y), \tau(u, x) \rangle$ is quasiconvex in $u \in X(x)$ for each fixed $(x, y) \in K \times C$ and X is convex-valued.

(iii) The condition that F is uniformly compact is unnecessary if C is compact.

The following corollaries are immediate.

Corollary 3.1.3 *Let $K \subset \mathbf{R}^n$ be a compact contractible absolute neighborhood retract and $C \subset \mathbf{R}^m$ be a closed contractible absolute neighborhood retract. Let X be a nonempty convex-valued continuous point-to-set mapping from K into itself and F a contractible-valued upper continuous and uniformly compact point-to-set mapping from K into C. Let $\theta : K \times C \longrightarrow \mathbf{R}^n$ be continuous. Suppose that there exists a compact contractible absolute neighborhood retract H such that $F(K) \subset H \subset C$. Then there exists a solution to the $GQVIP(X, F, \theta, K, C)$.* □

Corollary 3.1.4 *Let $K \subset \mathbf{R}^n$ be a compact contractible absolute neighborhood retract and $C \subset \mathbf{R}^m$ be a closed contractible absolute neighborhood retract. Let F be a contractible-valued upper continuous and uniformly compact point-to-set mapping from K into C. Let $\theta : K \times C \longrightarrow \mathbf{R}^n$ and $\tau : K \times K \longrightarrow \mathbf{R}^n$ be continuous. Suppose that*

(i) *there exists a compact contractible absolute neighborhood retract H such that $F(K) \subset H \subset C$,*

(ii) $\langle \theta(x,y), \tau(x,\cdot)\rangle \geq 0, \forall\ (x,y) \in K \times C$,

(iii) *for each fixed $(x,y) \in K \times C$, the set*

$$V(x,y) = \{u \in K : \langle \theta(x,y), \tau(u,x)\rangle = \min_{s\in K} \langle \theta(x,y), \tau(s,x)\rangle\}$$

is contractible.

Then there exists $x \in K$ that solves $GVIP(F, \theta, \tau, K, C)$. □

Corollary 3.1.5 *Let $K \subset \mathbf{R}^n$ be a compact contractible absolute neighborhood retract and $C \subset \mathbf{R}^m$ be a contractible absolute neighborhood retract. Let F be a closed contractible-valued upper continuous and uniformly compact point-to-set mapping from K into C. Let $\theta : K \times C \longrightarrow \mathbf{R}^n$ be continuous. Suppose that the set $Co(F(K)) \cap C$ is a retract of C. Then there exists $\bar{x} \in K$ that solves $GVIP(F, \theta, K, C)$.* □

In the case that K is unbounded, we have the following existence result for $GQVIP$.

Theorem 3.1.6 *Let K a be nonempty subset of $\mathbf{R}^n$ and C a nonempty closed convex subset of $\mathbf{R}^m$. Let X and F be point-to-set mappings from K into K and from K into C respectively. Let $\theta : K \times C \longrightarrow \mathbf{R}^n$ and $\tau : K \times K \longrightarrow \mathbf{R}^n$ be continuous single-valued functions. Suppose that there exists a nonempty compact convex subset B of K such that the following conditions hold:*

(i) $\tau(x,x) = 0, \forall\ x \in B$.

(ii) *F is contractible-valued upper continuous and uniformly compact on B,*

(iii) *$Y(x) = X(x) \cap B$ is a nonempty convex-valued continuous point-to-set mapping on B,*

(iv) *for each fixed $(x,y) \in K \times C$ $\langle \theta(x,y), \tau(u,x)\rangle$ is convex in $u \in Y(x)$,*

(v) *for all $x \in B$, $\mathrm{int}_{X(x)}(Y(x))$ is nonempty and for every $x \in \partial_{X(x)}(Y(x))$, there exists a $u \in \mathrm{int}_{X(x)}(Y(x))$ such that*

$$\langle \theta(x,y), \tau(u,x)\rangle \leq 0, \forall\ y \in F(x).$$

Then there exists a solution to the $GQVIP(X, F, \theta, \tau, K, C)$.

Proof. By Theorem 3.1.2 there exists $\bar{x} \in Y(\bar{x})$ and $\bar{y} \in F(\bar{x})$ such that

$$\langle \theta(\bar{x},\bar{y}), \tau(x,\bar{x})\rangle \geq 0, \forall\ x \in Y(\bar{x}). \tag{2}$$

Let $x \in X(\bar{x})$. There are two possibilities.

(i) $\bar{x} \in \mathrm{int}_{X(\bar{x})}(Y(\bar{x}))$. Then there exists $0 < \lambda < 1$ such that $\lambda x + (1-\lambda)\bar{x} \in Y(\bar{x})$. Then by (2) and (iv), we have

$$\begin{aligned} 0 &\leq \langle \theta(\bar{x},\bar{y}), \tau(\lambda x + (1-\lambda)\bar{x}, \bar{x})\rangle \\ &\leq \lambda\langle \theta(\bar{x},\bar{y}), \tau(x,\bar{x})\rangle + (1-\lambda)\langle \theta(\bar{x},\bar{y}), \tau(\bar{x},\bar{x})\rangle \\ &= \lambda\langle \theta(\bar{x},\bar{y}), \tau(x,\bar{x})\rangle \end{aligned}$$

Thus $\langle \theta(\bar{x},\bar{y}), \tau(x,\bar{x})\rangle \geq 0$.

(ii) $x \in \partial_{X(x)}(Y(x))$. By (v), there exists $u \in \text{int}_{X(\bar{x})}(Y(\bar{x}))$ such that

$$\langle \theta(\bar{x}, y), \tau(u, \bar{x}) \rangle \leq 0, \forall\ y \in F(\bar{x}).$$

In particular for $y = \bar{y}$, we have by (2) $\langle \theta(\bar{x}, \bar{y}), \tau(u, \bar{x}) \rangle = 0$. Now choose $0 < \lambda < 1$ such that $\lambda x + (1 - \lambda)u \in Y(x)$. Then we have

$$\begin{aligned} 0 &\leq \langle \theta(\bar{x}, \bar{y}), \tau(\lambda x + (1 - \lambda)u, \bar{x}) \rangle \\ &\leq \lambda \langle \theta(\bar{x}, \bar{y}), \tau(x, \bar{x}) \rangle + (1 - \lambda) \langle \theta(\bar{x}, \bar{y}), \tau(u, \bar{x}) \rangle \\ &= \lambda \langle \theta(\bar{x}, \bar{y}), \tau(x, \bar{x}) \rangle. \end{aligned}$$

So again $\langle \theta(\bar{x}, \bar{y}), \tau(x, \bar{x}) \rangle \geq 0$. Hence $\langle \theta(\bar{x}, \bar{y}), \tau(x, \bar{x}) \rangle \geq 0, \forall\ x \in X(\bar{x})$. Therefore $(\bar{x}, \bar{y})$ is a solution to the $GQVIP(X, F, \theta, \tau, K, C)$. □

We note that Theorem 3.1.6 extends a result due to Chan and Pang [5, Theorem 3.2]. By letting $X(x) = K$ for all $x \in K$, we have the following existence result for the $GVIP$.

Corollary 3.1.7 *Let K be nonempty subset of $\mathbf{R}^n$ and C be nonempty closed convex subset of $\mathbf{R}^m$. Let $F : K \longrightarrow C$ be a point-to-set mapping. Let $\theta : K \times C \longrightarrow \mathbf{R}^n$ and $\tau : K \times K \longrightarrow \mathbf{R}^n$ be continuous. Suppose that*

(i) $\tau(x, x) = 0, \forall\ x \in K$,

(ii) *there exists a compact convex subset $B \subset K$ with $\text{int}_K(B) \neq \emptyset$ such that F is upper continuous on B with $F(x)$ contractible and uniformly compact near x for all $x \in B$, and for each $x \in \partial_K(B)$, there exists $u \in \text{int}_K(B)$ such that $\langle \theta(x, y), \tau(u, x) \rangle \leq 0, \forall\ y \in F(x)$. Also $\langle \theta(x, y), \tau(u, x) \rangle$ is convex in $u \in B$ for each fixed $(x, y) \in K \times C$.*

Then there exists $x \in B$ which solves $GVIP(F, \theta, \tau, K, C)$. □

It is worth noting that we do not require F to be upper continuous, uniformly compact and contractible on K, and K need not be closed or bounded. The following corollary is immediate.

Corollary 3.1.8 *Assume that*

(i) *K is a nonempty (possibly unbounded and nonclosed) convex subset in $\mathbf{R}^n$,*

(ii) *F is a (possibly non-upper continuous) mapping from K into the family of subsets of $\mathbf{R}^n$,*

(iii) *there is a solid convex set E in $\mathbf{R}^n$, such that*

  (a) *$K \cap E$ is nonempty and compact,*

  (b) *F restricted to $K \cap E$ is upper continuous,*

  (c) *$F(x)$ is contractible and uniformly compact near x for each $x \in K \cap E$,*

  (d) *for each $x \in K \cap \partial(E)$, there is an $\bar{x} \in K \cap \text{int}(E)$, such that*

$$0 \leq \langle x - \bar{x}, y \rangle, \forall\ y \in F(x).$$

Then there is a solution (x, y) to $GVIP(K, F)$ with $\bar{x} \in E$.

Proof. Let $B = K \cap E$. Then B is a nonempty compact convex subset of K. First we claim that $K \cap \mathrm{int}(E) \subset \mathrm{int}_K(B)$ and $\partial_K(B) \subset K \cap \partial(E)$. Suppose $x \in K \cap \mathrm{int}(E)$. Then there exists an open set O such that $x \in O \subset E$ and $x \in K$. Then $A = K \cap O$ is open in K and $x \in A$. Since $A \subset B$. We have $A \subset \mathrm{int}_K(B)$. Therefore $x \in \mathrm{int}_K(B)$. Hence $K \cap \mathrm{int}(E) \subset \mathrm{int}_K(B)$. Next, suppose $x \in \partial_K(B)$. Let A be any neighborhood of x in K. Then $A \cap B \neq \emptyset$ and $A \cap (K \backslash B) \neq \emptyset$. Then $\emptyset \neq A \cap K \cap E \subset A \cap E$. Also

$$\begin{aligned} A \cap (K \backslash B) &= A \cap (K \cap (E^c \cup K^c)) \\ &= A \cap ((K \cap E^c) \cup (K \cap K^c)) \\ &= A \cap K \cap E^c. \end{aligned}$$

So $A \cap E^c \neq \emptyset$. If A is any neighborhood of x, then A is also a neighborhood of x in K. Thus by what we have shown, $A \cap E \neq \emptyset$ and $A \cap E^c \neq \emptyset$. Therefore $x \in K \cap \partial(E)$. Hence $\partial_K(B) \subset K \cap \partial(E)$. Now, let $x \in \partial_K(B)$. By (2) $x \in K \cap \partial(E)$. Then by (iii)(d), there is an $\bar{x} \in K \cap \mathrm{int}(E) \subset \mathrm{int}_K(B)$ such that

$$0 \leq \langle x - \bar{x}, y \rangle, \forall\, y \in F(x).$$

By letting $\theta(x, y) = y$ and $\tau(x, y) = x - y$, the condition (iii) of Corollary 3.1.7 is satisfied. Consequently, the result follows directly from Corollary 3.1.7. □

At first glance, it seems that the condition (iii) of Corollary 3.1.7 and those of Corollary 3.1.8 are the same. But in fact, the condition (iii) of Corollary 3.1.7 is actually weaker than those in Corollary 3.1.8. To see this, let us consider the following example. Let $K = \{(x, y) : \ y \geq 0\}$ and $E = \{(x, y) : x^2 + y^2 \leq 1,\ y > 0;\ -1 \leq x \leq 1,\ y = 0\}$. Then E is solid in $\mathbf{R}^2$ and we have

$$\begin{aligned} K \cap \partial(E) &= \{(x, y) : x^2 + y^2 = 1, y > 0; -1 \leq x \leq 1, y = 0\}, \\ \partial_K(K \cap E) &= \{(x, y) : x^2 + y^2 = 1, y \geq 0\}, \\ K \cap \mathrm{int}(E) &= \{(x, y) : x^2 + y^2 \leq 1, y > 0\}, \\ \mathrm{int}_K(K \cap E) &= \{(x, y) : x^2 + y^2 < 1, y > 0; -1 < x < 1, y = 0\}. \end{aligned}$$

It is easy to see that $\partial_K(K \cap E)$ is strictly contained in $K \cap \partial(E)$ and $K \cap \mathrm{int}(E)$ is also strictly contained in $\mathrm{int}_K(K \cap E)$.

For $r > 0$, let $B_r = \{x \in K : \|x\| \leq r\}$ and $C_r = \{x \in K : \|x\| = r\}$. The next corollary follows directly from Theorem 3.1.6.

Corollary 3.1.9 *Let K a be nonempty subset of $\mathbf{R}^n$ and C a nonempty closed convex subset of $\mathbf{R}^m$. Let X and F be point-to-set mappings from K into K and from K into C respectively. Let $\theta : K \times C \longrightarrow \mathbf{R}^n$ be continuous single-valued function. Suppose that there exists an $r > 0$ such that the following conditions hold:*

(i) *F is contractible-valued upper continuous and uniformly compact on B_r,*

(ii) *$Y(x) = X(x) \cap B_r$ is a nonempty convex-valued continuous point-to-set mapping on B_r,*

(iii) *for each* $x \in X(x) \cap C_r$, *there is a* $u \in X(x) \cap \text{int}_K(B_r)$ *such that*

$$\max_{y \in F(x)} \langle \theta(x, y), u - x \rangle \leq 0.$$

Then there exists a solution to the $GQVIP(X, F, \theta, K, C)$. $\square$

By following the same reasoning as in Theorem 3.1.6, we have the following existence result for $GQVIP(X, F, \theta, K, C)$.

Theorem 3.1.10 *Let* K *a be nonempty subset of* $\mathbf{R}^n$ *and* C *a nonempty closed convex subset of* $\mathbf{R}^m$. *Let* X *and* F *be point-to-set mappings from* K *into* K *and from* K *into* C *respectively. Let* $\theta : K \times C \longrightarrow \mathbf{R}^n$ *be continuous single-valued functions. Suppose that there exists a compact subset* B *of* K *such that the following conditions hold:*

(i) F *is contractible-valued upper continuous and uniformly compact on* B,

(ii) $Y(x) = X(x) \cap B$ *is a nonempty convex-valued continuous point-to-set mapping on* B,

(iii) *for each* $x \in B$ *and for each* $z \in X(x) \backslash B$, *there exists a vector* $u \in Y(x)$ *such that*

$$\max_{y \in F(x)} \langle \theta(x, y), u - z \rangle \leq 0.$$

Then there exists a solution to the $GQVIP(X, F, \theta, K, C)$.

Proof. By Corollary 3.1.3, there exist $\bar{x} \in Y(\bar{x})$ and $\bar{y} \in F(\bar{x})$ such that $\langle \theta(\bar{x}, \bar{y}), x - \bar{x} \rangle \geq 0$ for all $x \in Y(\bar{x})$. Now for $x \in X(\bar{x}) \backslash B$, by condition (iii), there exists a $u \in Y(\bar{x})$ such that $\langle \theta(\bar{x}, \bar{y}), x - u \rangle \geq 0$. On the other hand, we have $\langle \theta(\bar{x}, \bar{y}), u - \bar{x} \rangle \geq 0$. By adding the last two inequalities , we have $\langle \theta(\bar{x}, \bar{y}), x - \bar{x} \rangle \geq 0$. Hence $(\bar{x}, \bar{y})$ solves $GQVIP(X, F, \theta, K, C)$. $\square$

Remark. We note that the assertions due to Saigal [41, Lemma 4.1], Parida and Sen [37, Theorem 2], and Fang and Peterson [16, Theorem 3.2], respectively, may not be true in general by considering the example from Remark (iv) following Theorem 3.1.1.

3.2. Coercivity, Copositivity and Monotonicity

Normally, it is not easy to identify the compact set B in Theorem 3.1.6 or Corollary 3.1.9. We therefore consider some coercivity conditions on X and F that can be easily checked in some cases.

Theorem 3.2.1 *Let* K *be a nonempty subset of* $\mathbf{R}^n$ *and* C *a nonempty closed convex subset of* $\mathbf{R}^m$. *Let* X *and* F *be point-to-set mappings from* K *into* K *and from* K *into* C *respectively. Let* $\theta : K \times C \longrightarrow \mathbf{R}^n$ *and* $\tau : K \times K \longrightarrow \mathbf{R}^n$ *be continuous single-valued functions. Suppose that*

(i) $\tau(x, x) = 0, \forall\, x \in K$,

(ii) *for each fixed* $(x, y) \in K \times C$, $\langle \theta(x, y), \tau(u, x) \rangle$ *is convex in* $u \in X(x)$,

(iii) F *is contractible-valued upper continuous and uniformly compact on* K,

(iv) *there exists a vector* $x_0 \in \bigcap_{x \in K} X(x)$ *such that*

$$\lim_{\|x\| \longrightarrow \infty,\ x \in X(x)} \max_{y \in F(x)} \langle \theta(x,y), \tau(x_0, x) \rangle < 0,$$

(v) *there exists a* $\rho_0 > 0$ *such that* $X(x) \cap B_\rho$ *is a nonempty convex-valued continuous point-to-set mapping for all* $\rho \geq \rho_0$.

Then there exists a solution to the $GQVIP(X, F, \theta, \tau, K, C)$.

Proof. By condition (iv), there exists an $r_0 > 0$ such that for all $r \geq r_0$, if $x \in X(x) \cap C_r$, then $\max_{y \in F(x)} \langle \theta(x,y), \tau(x_0, x) \rangle \leq 0$. Now let $r > \max\{r_0, \|x_0\|, \rho_0\}$ and $B = B_r$. Then the condition (vi) of Theorem 3.1.6 is satisfied. Therefore the result follows from Theorem 3.1.6. □

The following corollary is immediate.

Corollary 3.2.2 *Let* K *be a nonempty subset of* $\mathbf{R}^n$ *and* C *a nonempty closed convex subset of* $\mathbf{R}^m$. *Let* X *and* F *be point-to-set mappings from* K *into* K *and from* K *into* C *respectively. Let* $\theta : K \times C \longrightarrow \mathbf{R}^n$ *be continuous single-valued function. Suppose that*

(i) F *is contractible-valued upper continuous and uniformly compact on* K,

(ii) *there exists a vector* $x_0 \in \bigcap_{x \in K} X(x)$ *such that*

$$\lim_{\|x\| \longrightarrow \infty, x \in X(x)} \max_{y \in F(x)} \langle \theta(x,y), x_0 - x \rangle < 0,$$

(iii) *there exists a* $\rho_0 > 0$ *such that* $X(x) \cap B_\rho$ *is a nonempty convex-valued continuous point-to-set mapping for all* $\rho \geq \rho_0$.

Then there exists a solution to the $GQVIP(X, F, \theta, K, C)$. □

The following corollary is a direct consequence of Corollary 3.2.2.

Corollary 3.2.3 *Let* K *be a nonempty subset in* $\mathbf{R}^n$ *and* $F : K \longrightarrow \mathbf{R}^n$ *a point-to-set mapping which is upper continuous, uniformly compact and contractible on* K. *Suppose there exists* $z \in K$ *such that*

$$\lim_{\|x\| \longrightarrow \infty,\ x \in K} (\inf_{y \in F(x)} \langle x - z, y \rangle) = \infty.$$

Then $GVIP(F, K)$ *has a solution.* □

Next, we have the following existence result for $GQVIP(X, F, \theta, K, C)$.

Theorem 3.2.4 *Let* K *be a nonempty subset of* $\mathbf{R}^n$ *and* C *a nonempty closed convex subset of* $\mathbf{R}^m$. *Let* X *and* F *be point-to-set mappings from* K *into* K *and from* K *into* C *respectively. Let* $\theta : K \times C \longrightarrow \mathbf{R}^n$ *be continuous single-valued function. Suppose that there exists an* $r > 0$ *and* $x_0 \in K$ *such that*

(i) $x_0 \in (\bigcap_{x \in C_r} X(x)) \cap \mathrm{int}_K(B_r)$ *and*

$$\inf_{x \in X(x) \cap C_r} \inf_{y \in F(x)} \langle \theta(x,y), x - x_0 \rangle \geq 0,$$

(ii) *F is contractible-valued upper continuous and uniformly compact on B_r,*

(iii) *$Y(x) = X(x) \cap B_r$ is a nonempty convex-valued continuous point-to-set mapping on B_r.*

Then there exists a solution to the GQVIP(X, F, θ, K, C).

Proof. By letting $B = B_r$, it suffices to verify that the condition (iii) of Corollary 3.1.9 is satisfied. If there is no vector x such that $x \in \partial_{X(x)}(Y(x))$, then clearly the condition (iii) of Corollary 3.1.9 is satisfied. On the other hand, for each $x \in \partial_{X(x)}(Y(x))$, we have $x_0 \in X(x) \cap \text{int}_K(B_r) \subset \text{int}_{X(x)}(Y(x))$ and $\max_{y \in F(x)} \langle \theta(x, y), x_0 - x \rangle \leq 0$. Therefore the condition (iii) of Corollary 3.1.9 is satisfied. Hence the result follows from Corollary 3.1.9. □

Now we derive some existence results under certain monotonicity and copositivity conditions. First, let us introduce the following definitions.

Definition 3.2.5 Let X and F be two point-to-set mappings on a set K.

(i) F is said to be *monotone with respect to* X on K, if for any $x_1 \in X(x_1)$ and $x_2 \in X(x_2)$, we have

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0, \forall\ y_1 \in F(x_1), y_2 \in F(x_2).$$

(ii) F is said to be *strictly monotone with respect to* X on K, if for any $x_1 \in X(x_1)$ and $x_2 \in X(x_2)$ with $x_1 \neq x_2$, we have

$$\langle y_1 - y_2, x_1 - x_2 \rangle > 0, \forall\ y_1 \in F(x_1), y_2 \in F(x_2).$$

(iii) F is said to be *strongly monotone with respect to* X on K if there exists a scalar $\alpha > 0$ such that for any $x_1 \in X(x_1)$ and $x_2 \in X(x_2)$, we have

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq \alpha \|x_1 - x_2\|^2, \forall\ y_1 \in F(x_1), y_2 \in F(x_2).$$

(iv) F is said to be *b-monotone with respect to* X on K if there exists an increasing function $b : [0, \infty) \longrightarrow [0, \infty)$ with $b(0) = 0$ and $b(r) \longrightarrow \infty$ as $r \longrightarrow \infty$ such that for any $x_1 \in X(x_1)$ and $x_2 \in X(x_2)$, we have

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq \|x_1 - x_2\| b(\|x_1 - x_2\|), \forall\ y_1 \in F(x_1), y_2 \in F(x_2).$$

(v) F is said to be *copositive with respect to* X *at the point* x_0 on K if $x_0 \in X(x_0)$ and there exists a $y_0 \in F(x_0)$ such that for all $x \in K$ with $x \in X(x)$, we have

$$\langle y - y_0, x - x_0 \rangle \geq 0, \forall\ y \in F(x).$$

(vi) F is said to be *strictly copositive with respect to* X *at the point* x_0 on K if $x_0 \in X(x_0)$ and there exists a $y_0 \in F(x_0)$ such that for all $x \in K$ with $x \in X(x), x \neq x_0$, we have

$$\langle y - y_0, x - x_0 \rangle > 0, \forall\ y \in F(x).$$

(vii) F is said to be *strongly copositive with respect to* X *at the point* x_0 on K if $x_0 \in X(x_0)$ and there exists a scalar $\alpha > 0$ and a $y_0 \in F(x_0)$ such that for all $x \in K$ with $x \in X(x)$, we have

$$\langle y - y_0, x - x_0 \rangle \geq \alpha \|x - x_0\|^2, \forall\, y \in F(x).$$

(viii) F is said to be *b-copositive with respect to* X *at the point* x_0 on K if there exists an increasing function $b : [0, \infty) \longrightarrow [0, \infty)$ with $b(0) = 0$ and $b(r) \longrightarrow \infty$ as $r \longrightarrow \infty$, and if $x_0 \in X(x_0)$ and there exists a $y_0 \in F(x_0)$ such that for all $x \in K$ with $x \in X(x)$, we have

$$\langle y - y_0, x - x_0 \rangle \geq \|x - x_0\| b(\|x - x_0\|), \forall\, y \in F(x).$$

Remark. If $X(x) = K$ for all $x \in K$ and $x_0 = 0$, then (i), (ii), (iii), (v), (vi), (vii) of Definition 3.2.5 reduce to the usual definitions ([41, Definition 3.1, 3.2]) of monotonicity and copositivity for point-to-set mappings. If $K = \mathbf{R}^n$, then the above definitions of strong copositivity and strong monotonicity reduce to the ones introduced by Chan and Pang [5]. Clearly, if $F(x_0)$ is nonempty and if $x_0 \in X(x_0)$, then monotonicity, strict monotonicity, strong monotonicity and b-monotonicity imply copositivity, strict copositivity, strong copositivity and b-copositivity respectively.

The following gives an existence result for the $GQVIP(X, F, \theta, K, C)$ under the strong copositivity condition.

Theorem 3.2.6 *Let* K *be a nonempty subset of* $\mathbf{R}^n$ *and* C *a nonempty closed convex subset of* $\mathbf{R}^m$. *Let* X *and* F *be point-to-set mappings from* K *into* K *and from* K *into* C *respectively. Let* $\theta : K \times C \longrightarrow \mathbf{R}^n$ *be continuous single-valued function. Suppose that*

(i) *there exists* $x_0 \in \bigcap_{x \in K} X(x)$ *such that the point-to-set mapping*

$$V(x) = \{\theta(x, y) : y \in F(x)\}$$

is either b-copositive or strongly copositive with respect to X *at* x_0 *on* K,

(ii) F *is contractible-valued upper continuous and uniformly compact on* K,

(iii) *there exists a* $\rho_0 > 0$ *such that* $X(x) \cap B_\rho$ *is a nonempty convex-valued continuous point-to-set mapping for all* $\rho \geq \rho_0$.

Then there exists a solution to the $GQVIP(X, F, \theta, K, C)$.

Proof. Since strong copositivity implies b-copositivity, it suffices to prove this theorem under the assumption that V is b-copositive with respect to X at the point x_0. Then there exist an increasing function $b : [0, \infty) \longrightarrow [0, \infty)$ with $b(0) = 0$ and $b(r) \longrightarrow \infty$ as $r \longrightarrow \infty$ and a $y_0 \in F(x_0)$ such that for all $x \in K$ with $x \in X(x)$, we have

$$\langle y - y_0, x - x_0 \rangle \geq \|x - x_0\| b(\|x - x_0\|), \forall\, y \in F(x).$$

Then we have for all $y \in F(x)$,

$$\begin{aligned} \langle \theta(x, y), x_0 - x \rangle &\leq -\|x - x_0\| b(\|x - x_0\|) + \langle \theta(x_0, y_0), x_0 - x \rangle \\ &\leq -\|x_0 - x\| (b(\|x_0 - x\|) - \|\theta(x_0, y_0)\|). \end{aligned}$$

Since $b(r) \longrightarrow \infty$ as $r \longrightarrow \infty$, we then have

$$\lim_{\|x\| \longrightarrow \infty, x \in X(x)} \max_{y \in F(x)} \langle \theta(x, y), x_0 - x \rangle = -\infty.$$

Thus the condition (ii) of Corollary 3.2.2 is satisfied. Hence the result follows from Corollary 3.2.2. □

The following corollary is immediate.

Corollary 3.2.7 *Let K be a nonempty subset of $\mathbf{R}^n$ and C a nonempty closed convex subset of $\mathbf{R}^m$. Let X and F be point-to-set mappings from K into K and from K into C, respectively. Let $\theta : K \times C \longrightarrow \mathbf{R}^n$ be a continuous single-valued function. Suppose that*

(i) *the set $\bigcap_{x \in K} X(x)$ is not empty and the point-to-set mapping*

$$V(x) = \{\theta(x, y) : y \in F(x)\}$$

is either b-monotone or strongly monotone with respect to X on K,

(ii) *F is contractible-valued upper continuous and uniformly compact on K,*

(iii) *there exists a $\rho_0 > 0$ such that $X(x) \bigcap B_\rho$ is a nonempty convex-valued continuous point-to-set mapping for all $\rho \geq \rho_0$.*

Then there exists a solution to the $GQVIP(X, F, \theta, K, C)$. □

Remark. In general, solutions in Theorem 3.2.6 and Corollary 3.2.7 are not unique. To see this, consider the following example. Let $K = C = \mathbf{R}^n$. For any $x \in \mathbf{R}^n$, let $X(x) = \{\alpha x : 0 \leq \alpha \leq 1\}$ and $F(x) = \{2x\}$. Let $\theta(x, y) = y - x$. Thus condition (iii) of Theorem 3.2.6 is satisfied. We claim that X is continuous. To see this, assume that the sequence $\{x_n\}$ converges to x and the sequence $\{y_n\}$ converges to y with $y_n \in X(x_n)$ for all n. Then for each n, $y_n = \alpha_n x_n$ for some $0 \leq \alpha_n \leq 1$. It is clear that $\{\alpha_n\}$ has a convergent subsequence. Without loss of generality, we may assume that $\{\alpha_n\}$ is convergent with limit α. Clear $0 \leq \alpha \leq 1$. Then by letting n approach ∞, we get $y = \alpha x$. Thus $y \in X(x)$. Hence X is upper continuous. On the other hand, suppose that $y \in X(x)$ and x_n converges to x. Then $y = \alpha x$ for some $0 \leq \alpha \leq 1$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that α_n converges to α and let $y_n = \alpha_n x_n$ for all n. Then $y_n \in X(x_n)$ for all n and y_n converges to y. Hence X is also lower continuous. Consequently, X is continuous as claimed. It is clear that $X|B_\rho$ is also continuous for all $\rho > 0$. Also it is easy to see that F is b-copositive, strongly copositive (at the point 0), b-monotone and strongly monotone with respect to X on $\mathbf{R}^n$. But it is clear that *every* vector in $\mathbf{R}^n$ is a solution to the $GQVIP(X, F, \theta, \mathbf{R}^n, \mathbf{R}^n)$. Hence solutions are not unique in this case. However, as the following two corollaries show, the solutions are unique if we assume that X is a constant point-to-set mapping with $X(x) = K$ for all $x \in K$.

Corollary 3.2.8 *Let K be a nonempty subset of $\mathbf{R}^n$ and C a nonempty closed convex subset of $\mathbf{R}^m$. Let F be a point-to-set mapping from K into C. Let $\theta : K \times C \longrightarrow \mathbf{R}^n$ be continuous single-valued function. Suppose that*

(i) *there exists* $x_0 \in \bigcap_{x\in K} X(x)$ *such that the point-to-set mapping*

$$V(x) = \{\theta(x,y) : y \in F(x)\}$$

is either b-copositive or strongly copositive with respect to x_0 *on* K,

(ii) F *is contractible-valued upper continuous and uniformly compact on* K.

Then the problem $GVIP(F,\theta,K,C)$ *has a unique solution.*

Proof. It suffices to prove this corollary under the assumption that V is b-copositive with respect to the point x_0 on K. Suppose that x_1 and x_2 are both solutions to the $GVIP(F,\theta,K,C)$. Then there exist $y_i \in F(x_i), i = 1,2$ such that

$$\langle \theta(x_i, y_i), u - x_i\rangle \geq 0, \forall\ u \in K, i = 1,2. \tag{3}$$

From (3), we have

$$\langle \theta(x_1,y_1) - \theta(x_2,y_2), x_1 - x_2\rangle \leq 0. \tag{4}$$

On the other hand, since V is b-copositive, there exist an increasing function $b : [0,\infty) \longrightarrow [0,\infty)$ with $b(0) = 0$ and $b(r) \longrightarrow \infty$ as $r \longrightarrow \infty$ and a $y_0 \in F(x_0)$ such that for all $i = 1,2$, we have

$$\langle \theta(x_i,y_i) - \theta(x_0,y_0), x_i - x_0\rangle \geq ||x_i - x_0||b(||x_i - x_0||). \tag{5}$$

From (5), we have

$$\langle \theta(x_1,y_1) - \theta(x_2,y_2), x_1 - x_2\rangle \geq \sum_{i=1}^{2} ||x_i - x_0||b(||x_i - x_0||). \tag{6}$$

Combining (4) and (6), we then have $x_1 = x_2 = x_0$. Hence the $GVIP(F,\theta,K,C)$ has a unique solution. □

The following corollary is a direct consequence of Corollaries 3.2.7 and 3.2.8.

Corollary 3.2.9 *Let* K *be a nonempty subset of* $\mathbf{R}^n$ *and* C *a nonempty closed convex subset of* $\mathbf{R}^m$. *Let* F *be a point-to-set mapping from* K *into* C. *Let* $\theta : K \times C \longrightarrow \mathbf{R}^n$ *be continuous single-valued function. Suppose that*

(i) *the set* $\bigcap_{x\in K} X(x)$ *is not empty and the point-to-set mapping*

$$V(x) = \{\theta(x,y) : y \in F(x)\}$$

is either b-monotone or strongly monotone with respect to x_0 *on* K,

(ii) F *is contractible-valued upper continuous and uniformly compact on* K.

Then the prol . n $GVIP(F, \theta, K, C)$ has a unique solution. □

We note that by letting $\theta(x, y) = y$ and $C = \mathbf{R}^n$, Corollary 3.2.9 extends a similar result due to Saigal ([41, Theorem 3.1]) where he did not consider the uniqueness of the solution of the $GVIP(F, K)$.

Let $\tau : K \times K \longrightarrow \mathbf{R}^n$ be a single-valued function such that $\tau(x, x) = 0$ for all $x \in K$. F is said to be *τ-monotone with respect to* X on K if for any $x_1 \in X(x_1)$ and $x_2 \in X(x_2)$, we have

$$\langle y_1, \tau(x_2, x_1)\rangle + \langle y_2, \tau(x_1, x_2)\rangle \leq 0, \forall\ y_1 \in F(x_1), y_2 \in F(x_2).$$

An example of $\tau-$monotone point-to-set mapping is the following. Let $K = C = \mathbf{R}$ and

$$\begin{aligned} \tau(x, y) &= e^x - e^y \\ X(x) &= \{\alpha x : 0 \leq \alpha \leq 1\} \\ F(x) &= \{2x\}. \end{aligned}$$

Then F is $\tau-$monotone with respect to X on $\mathbf{R}$. If $X(x) = K$ for all $x \in K$, then this definition reduces to the definition of τ-monotone point-to-set mappings introduced by Parida and Sen [37].

Theorem 3.2.10 *Let K be a nonempty subset of $\mathbf{R}^n$ and C a nonempty closed convex subset of $\mathbf{R}^m$. Let X and F be point-to-set mappings from K into K and from K into C, respectively. Let $\theta : K \times C \longrightarrow \mathbf{R}^n$ and $\tau : K \times K \longrightarrow \mathbf{R}^n$ be continuous single-valued functions. Suppose that*

(i) $\tau(x, x) = 0, \forall\ x \in K$,

(ii) *for each fixed* $(x, y) \in K \times C$, $\langle \theta(x, y), \tau(u, x)\rangle$ *is convex in* $u \in X(x)$,

(iii) F *is contractible-valued upper continuous and uniformly compact on* K,

(iv) *the point-to-set mapping* $V(x) = \{\theta(x, y) : y \in F(x)\}$ *is τ-monotone with respect to* X *on* K,

(v) *there exist vectors* $x_0 \in \bigcap_{x \in K} X(x)$ *and* $y_0 \in F(x_0)$ *such that*

$$\lim_{\|x\| \longrightarrow \infty,\ x \in X(x)} \langle \theta(x_0, y_0), \tau(x, x_0)\rangle > 0,$$

(vi) *there exists a* $\rho_0 > 0$ *such that* $X(x) \cap B_\rho$ *is a nonempty convex-valued continuous point-to-set mapping for all* $\rho \geq \rho_0$.

Then there exists a solution to the $GQVIP(X, F, \theta, \tau, K, C)$.

Proof. By condition (v), there exists an $r_0 > 0$ such that for all $r \geq r_0$, if $x \in X(x) \cap C_r$, then $\langle \theta(x_0, y_0), \tau(x, x_0)\rangle > 0$. Thus for such x, since $x_0 \in X(x_0), x \in X(x)$ and V is τ-monotone with respect to X on K, we have

$$\langle \theta(x, y), \tau(x_0, x)\rangle \leq -\langle \theta(x_0, y_0), \tau(x, x_0)\rangle < 0,\ \forall\ y \in F(x).$$

Let $r \geq \max\{r_0, \|x_0\|, \rho_0\}$ and $B = B_r$. Then the condition (vi) of Theorem 3.1.6 is satisfied. Hence the result follows from Theorem 3.1.6. □

Given two point-to-set mappings F and X on K, F is said to be *pseudo-monotone with respect to* X on K if for any $x_1 \in X(x_1), x_2 \in X(x_2)$ and $y_1 \in F(x_1), y_2 \in F(x_2)$, $\langle x_1 - x_2, y_2 \rangle \geq 0$ implies $\langle x_1 - x_2, y_1 \rangle \geq 0$.

Theorem 3.2.11 *Let K be a nonempty subset of $\mathbf{R}^n$ and C be a nonempty closed convex subset of $\mathbf{R}^m$. Let X and F be point-to-set mappings from K into K and from K into C respectively. Let $\theta : K \times C \longrightarrow \mathbf{R}^n$ be continuous single-valued function. Suppose that*

(i) *F is contractible-valued upper continuous and uniformly compact on K,*

(ii) *the point-to-set mapping $V(x) = \{\theta(x, y) : y \in F(x)\}$ is pseudo-monotone with respect to X on K,*

(iii) *there exist vectors $x_0 \in \bigcap_{x \in K} X(x)$ and $y_0 \in F(x_0)$ such that $\theta(x_0, y_0) \in \mathrm{int}(\bigcup_{x \in K} X(x))^*$,*

(iv) *there exists a $\rho_0 > 0$ such that $X(x) \cap B_\rho$ is a nonempty convex-valued continuous point-to-set mapping for all $\rho \geq \rho_0$.*

Then there exists a solution to the $GQVIP(X, F, \theta, K, C)$.

Proof. Let $S = \{x \in X(x) : \langle \theta(x_0, y_0), x - x_0 \rangle \leq 0\}$. We claim that S is compact. Clearly S is closed. Suppose that S is unbounded. Then there exists a sequence $\{x_n\} \subset S$ such that $\|x_n\| \longrightarrow \infty$. Let $x^n = x_n / \|x_n\|$. Then the sequence $\{x^n\}$ is bounded and thus has a convergent subsequence. Without loss of generality, we may assume that the entire sequence $\{x^n\}$ converges to a vector e. Clearly $\|e\| = 1$. For each n, we have

$$0 \leq \langle \theta(x_0, y_0), x^n \rangle \leq \langle \theta(x_0, y_0), x_0 / \|x_n\| \rangle.$$

By passing to the limit, we obtain $\langle \theta(x_0, y_0), e \rangle = 0$. By (iii), there exists an $\epsilon > 0$ such that $\theta(x_0, y_0) - \epsilon e \in (\bigcup_{x \in K} X(x))^*$. Then for each n, we have

$$\langle \theta(x_0, y_0) - \epsilon e, x^n \rangle \geq 0.$$

By passing to the limit, we have

$$0 \leq \langle \theta(x_0, y_0) - \epsilon e, e \rangle = -\epsilon < 0$$

which is a contradiction. Hence S is bounded and thus S is compact as claimed. Now choose $\rho > \rho_0$ such that $S \subset \mathrm{int}_K(B_\rho)$. All the conditions of Corollary 3.1.9, except (iii) are satisfied. To show that the condition (iii) is also satisfied, let $x \in X(x) \cap B_\rho$. Then $x \notin S$ which implies $\langle \theta(x_0, y_0), x - x_0 \rangle > 0$. By the pseudo-monotonicity of V, we have

$$\langle \theta(x, y), x - x_0 \rangle \geq 0, \forall\ y \in F(x).$$

Therefore, the $GQVIP(X, F, \theta, K, C)$ has a solution by Corollary 3.1.9. □

The following corollaries are immediate.

Corollary 3.2.12 *Let K be a nonempty subset of $\mathbf{R}^n$ and C a nonempty closed convex subset of $\mathbf{R}^m$. Let X and F be point-to-set mappings from K into K and from K into C respectively. Let $\theta : K \times C \longrightarrow \mathbf{R}^n$ be a continuous single-valued function. Suppose that*

(i) F *is contractible-valued upper continuous and uniformly compact on* K,

(ii) *the point-to-set mapping* $V(x) = \{\theta(x,y) : y \in F(x)\}$ *is monotone with respect to* X *on* K,

(iii) *there exist vectors* $x_0 \in \bigcap_{x \in K} X(x)$ *and* $y_0 \in F(x_0)$ *such that*

$$\theta(x_0, y_0) \in \mathrm{int}(\bigcup_{x \in K} X(x))^*,$$

(iv) *there exists a* $\rho_0 > 0$ *such that* $X(x) \cap B_\rho$ *is a nonempty convex-valued continuous point-to-set mapping for all* $\rho \geq \rho_0$.

Then there exists a solution to the $GQVIP(X, F, \theta, K, C)$. □

Corollary 3.2.13 *Let* K *be a nonempty subset of* $\mathbf{R}^n$ *and* C *be a nonempty closed convex subset of* $\mathbf{R}^m$. *Let* F *be a point-to-set mapping from* K *into* C *and let* $\theta : K \times C \longrightarrow \mathbf{R}^n$ *be a continuous single-valued function. Suppose that*

(i) F *is nonempty contractible valued upper continuous and uniformly compact on* K,

(ii) *the point to-set-mapping* $V(x) = \{\theta(x,y) : y \in F(x)\}$ *is monotone or pseudo-monotone on* K,

(iii) *there exist vectors* $x_0 \in K$ *and* $y_0 \in F(x_0)$ *such that* $\theta(x_0, y_0) \in \mathrm{int}(K^*)$.

Then there exists a solution to the $GVIP(F, \theta, K, C)$. □

Next, we present an existence result for the $GQVIP$ which does not employ Theorem 3.1.6.

Theorem 3.2.14 *Let* K *and* C *be nonempty closed convex subsets of* $\mathbf{R}^n$ *and* $\mathbf{R}^m$, *respectively. Let* X *and* F *be point-to-set mappings from* K *into* K *and from* K *into* C, *respectively. Let* $\theta : K \times C \longrightarrow \mathbf{R}^n$ *and* $\tau : K \times K \longrightarrow \mathbf{R}^n$ *be continuous single-valued functions. Suppose that*

(i) $\tau(x, x) = 0, \forall\, x \in K$,

(ii) *for each fixed* $(x, y) \in K \times C, \langle \theta(x,y), \tau(u,x) \rangle$ *is quasiconvex in* $u \in X(x)$,

(iii) F *is contractible-valued upper continuous and uniformly compact on* K,

(iv) *there exists a* $\rho_0 > 0$ *such that* $X_\rho(x) = X(x) \cap B_\rho$ *is a nonempty convex-valued continuous point-to-set mapping for all* $\rho \geq \rho_0$.

Then

(v) *there exists* $x_\rho \in X_\rho(x_\rho)$ *that solves the* $GQVIP(X_\rho, F, \theta, \tau, B_\rho, C)$ *for each* $\rho \geq \rho_0$.

(vi) *if the set* $\{x_\rho\}$ *has a convergent subsequence, then* $GQVIP(X, F, \theta, \tau, K, C)$ *has a solution.*

Proof. The result (v) follows directly from Theorem 3.1.2. Suppose that the set $\{x_\rho\}$ has a convergent subsequence $\{x_n\}$ with limit x_0 and $x_n \in X_{\rho_n}(x_n)$ for all n. Then for each n, there exists $y_n \in F(x_n)$ such that

$$\langle \theta(x_n, y_n), \tau(x, x_n)\rangle \geq 0, \forall\ x \in X_{\rho_n}(x_n).$$

Clearly $x_0 \in X(x_0)$ and $\{y_n\}$ has a convergent subsequence. Without loss of generality, we may assume that the entire sequence $\{y_n\}$ converges to a limit y_0. Then $y_0 \in F(x_0)$. For each $x \in X(x_0)$, there is an m such that $x \in B_{\rho_m}$. Then $x \in X_{\rho_n}(x_0)$ for all $\rho_n \geq \rho_m$. Since X is continuous, there exist k and z_n such that $z_n \longrightarrow x$ and $z_n \in X(x_n)$ for all $n \geq k$. Also there exists an ℓ such that $z_n \in B_{\rho_n}$ for all $n \geq \ell$ since we can choose ρ_m large enough such that $x \in \text{int}(B_{\rho_m})$. Then for all $n \geq \max\{m, k, \ell\}$, we have $z_n \in X_{\rho_n}(x_n)$ and

$$\langle \theta(x_n, y_n), \tau(z_n, x_n)\rangle \geq 0.$$

By passing to the limit, we obtain the inequality $\langle \theta(x_0, y_0), \tau(z, x_0)\rangle \geq 0$. Therefore (x_0, y_0) is a solution to $GQVIP(X, F, \theta, \tau, K, C)$. □

The following corollary is immediate.

Corollary 3.2.15 *Under the assumptions of Theorem 3.2.14, it follows that*

(i) *there exists $x_\rho \in X_\rho(x_\rho)$ that solves the $GQVIP(X_\rho, F, \theta, \tau, B_\rho, C)$ for each $\rho \geq \rho_0$.*

(ii) *if the set $\{x_\rho\}$ is bounded, then the $GQVIP(X, F, \theta, \tau, K, C)$ has a solution.* □

3.3. Analysis of the Solution Set of GQVIP

In this subsection we shall discuss various properties of solution sets of generalized quasi-variational inequality problems. Though such results are important in sensitivity analysis, very few results have been seen in the literature. See Hartman and Stampacchia [20], McLinden [28, 29] and Fang and Peterson [16]. As a matter of fact, to determine whether the solution set of a GQVIP possesses some interesting properties, such as compactness and convexity, is a fairly difficult task. Our first result is that the solution set of any GQVIP is always closed under fairly general conditions.

Theorem 3.3.1 *Let K and C be nonempty subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. Let $X : K \longrightarrow K$ and $F : K \longrightarrow C$ be point-to-set mappings. Also let $\theta : K \times C \longrightarrow \mathbf{R}^n$ and $\tau : K \times K \longrightarrow \mathbf{R}^n$ be continuous single-valued functions. Suppose X is continuous on K and that F is upper continuous and uniformly compact on K. Then the solution set S of $GQVIP(X, F, \theta, \tau, K, C)$ is closed.*

Proof. The result is clearly true if S is empty. So let us suppose that S is not empty. Let x be a limit point of S. Then there exists a sequence $\{x_n\} \subset S$ such that x_n converges to x. For each n, since $x_n \in S$, $x_n \in X(x_n)$ and there exists $y_n \in F(x_n)$ such that

$$\langle \theta(x_n, y_n), \tau(u, x_n)\rangle \geq 0, \forall\ u \in X(x_n). \tag{7}$$

Clearly $x \in X(x)$. Since F is upper continuous and uniformly compact, $\{y_n\}$ has a convergent subsequence. Without loss of generality, we may assume that $\{y_n\}$ converges to y. Then $y \in F(x)$ by the upper continuity of F. Now for any $u \in X(x)$, since X is continuous, there exist n_0 and z_n such that $z_n \in X(x_n)$ for all $n \geq n_0$ and z_n converges to u. By (7), we have

$$\langle \theta(x_n, y_n), \tau(z_n, x_n) \rangle \geq 0, \forall\ n \geq n_0.$$

Passing to the limit, we have, since θ and τ are continuous,

$$\langle \theta(x, y), \tau(u, x) \rangle \geq 0.$$

Consequently, $x \in S$. Since S contains all of its limit points, it is closed. □

Under some conditions, the solution set of a $GQVIP$ can be shown to be compact as the following theorem illustrates.

Theorem 3.3.2 *Let K and C be nonempty subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. Let $X : K \longrightarrow K$ and $F : K \longrightarrow C$ be point-to-set mappings. Also let $\theta : K \times C \longrightarrow \mathbf{R}^n$ and $\tau : K \times K \longrightarrow \mathbf{R}^n$ be continuous single-valued functions. Suppose X is continuous on K and that F is upper continuous and uniformly compact on K. Suppose there exists a vector $x_0 \in \bigcap_{x \in K} X(x)$ such that*

$$\lim_{\|x\| \longrightarrow \infty,\ x \in X(x)} \max_{y \in F(x)} \langle \theta(x, y), \tau(x_0, x) \rangle = -\infty.$$

Then the solution set S of $GQVIP(X, F, \theta, \tau, K, C)$ is compact.

Proof. The result is clearly true if S is empty. Suppose that S is not empty. The closedness of S follows from Theorem 3.3.1. By assumption, there exists an $r > 0$ such that for all $\|x\| > r$ with $x \in X(x)$ we have

$$\max_{y \in F(x)} \langle \theta(x, y), \tau(x_0, x) \rangle < 0.$$

It then follows that $S \subseteq B_r$. Consequently S is bounded and hence compact. □

As we indicated in the remark following Corollary 3.2.7, even the point-to-set mapping $V(x) = \{\theta(x, y) : y \in F(x)\}$ is strongly monotone with respect to the point-to-set mapping X on K, the $GQVIP(X, F, \theta, K, C)$ does not necessarily have a unique solution. Nevertheless, the solution set is necessarily compact as the following corollary shows.

Corollary 3.3.3 *Let K and C be nonempty subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. Let X and F be point-to-set mappings from K into K and C, respectively. Also let $\theta : K \times C \longrightarrow \mathbf{R}^n$ and $\tau : K \times K \longrightarrow \mathbf{R}^n$ be continuous single-valued functions. Suppose X is continuous on K and F is upper continuous and uniformly compact on K. Then the solution set S of $GQVIP(X, F, \theta, K, C)$ is compact if one of the following conditions holds:*

(i) there exists $x_0 \in \bigcap_{x \in K} X(x)$ such that the point-to-set mapping

$$V(x) = \{\theta(x, y) : y \in F(x)\}$$

is either b-copositive or strongly copositive with respect to x_0 on K,

(ii) *the set $\bigcap_{x\in K} X(x)$ is not empty and the point-to-set mapping $V(x)$ is either b-monotone or strongly monotone with respect to X on K.*

Proof. It is clear that it suffices to prove this corollary under the assumption that V is b-copositive with respect to X at x_0 on K. Then there exist an increasing function $b : [0,\infty) \longrightarrow [0,\infty)$ with $b(0) = 0$ and $b(r) \longrightarrow \infty$ as $r \longrightarrow \infty$, and a $y_0 \in F(x_0)$ such that for all $x \in K$ with $x \in X(x)$, we have

$$\langle \theta(x,y) - \theta(x_0,y_0), x - x_0\rangle \geq \|x - x_0\| b(\|x - x_0\|), \forall\ y \in F(x).$$

Consequently, we have

$$\begin{aligned}
\max_{y\in F(x)} \langle \theta(x,y), x_0 - x\rangle &= -\min_{y\in F(x)} \langle \theta(x,y), x - x_0\rangle \\
&\leq \langle \theta(x_0,y_0), x_0 - x\rangle - \|x - x_0\| b(\|x - x_0\|) \\
&\leq -\|x - x_0\| (b(\|x - x_0\|) - \|\theta(x_0,y_0\|).
\end{aligned}$$

Therefore, we have

$$\lim_{\|x\|\longrightarrow\infty, x\in X(x)} \max_{y\in F(x)} \langle \theta(x,y), x_0 - x\rangle = -\infty.$$

Hence by Theorem 3.3.2, S is compact. □

In the case that X is the constant mapping K, we can further strengthen the result of Corollary 3.3.3.

Theorem 3.3.4 *Let K be a nonempty subset of $\mathbf{R}^n$ and C a nonempty closed convex subset of $\mathbf{R}^m$. Let F be a point-to-set mapping from K into C and $\theta : K \times C \longrightarrow \mathbf{R}^n$ be a continuous single-valued function. Then the solution set S of $GVIP(F,\theta,K,C)$ is either empty or a singleton if one of the following conditions holds:*

(i) *there exists $x_0 \in \bigcap_{x\in K} X(x)$ such that the point-to-set mapping*

$$V(x) = \{\theta(x,y) : y \in F(x)\}$$

is strictly copositive, b-copositive or strongly copositive with respect to x_0 on K,

(ii) *the set $\bigcap_{x\in K} X(x)$ is not empty and the point-to-set mapping $V(x)$ is strictly monotone, b-monotone, or strongly monotone on K.*

Proof. It suffices to prove this theorem under the condition that V is strictly copositive with respect to x_0 on K. If S is empty, then we have nothing to prove. Otherwise suppose that $x_1, x_2 \in S$ with $x_1 \neq x_2$. There are two cases to be discussed:

(a): Both x_1 and x_2 are not equal to x_0. Then as in the proof of Corollary 3.2.8, there exist $y_i \in F(x_i), i = 1,2$ such that

$$\langle \theta(x_1,y_1) - \theta(x_2,y_2), x_1 - x_2\rangle \leq 0. \tag{8}$$

On the other hand, since V is strictly copositive, there exists $y_0 \in F(x_0)$ such that

$$\langle \theta(x_i, y_i) - \theta(x_0, y_0), x_i - x_0 \rangle > 0, \ i = 1, 2. \tag{9}$$

From (9), we have

$$\langle \theta(x_1, y_1) - \theta(x_2, y_2), x_1 - x_2 \rangle > 0$$

which contradicts (8).

(b): Either x_1 or x_2 equals x_0, say $x_2 = x_0$. Then there exists $y_1 \in F(x_1)$ and $y_0' \in F(x_0)$ such that

$$\langle \theta(x_1, y_1), x_0 - x_1 \rangle \geq 0 \tag{10}$$

and

$$\langle \theta(x_0, y_0'), x_1 - x_0 \rangle \geq 0. \tag{11}$$

From (10) and (11), we have

$$\langle \theta(x_1, y_1) - \theta(x_0, y_0'), x_1 - x_0 \rangle \leq 0. \tag{12}$$

On the other hand, there exists $y_0 \in F(x_0)$ such that

$$\langle \theta(x_1, y_1) - \theta(x_0, y_0), x_1 - x_0 \rangle > 0. \tag{13}$$

Note that y_0 is not necessarily equal to y_0'. Since $\langle \theta(x_0, y_0') - \theta(x_0, y_0), 0 \rangle = 0$, combining this with (13), we have

$$\langle \theta(x_1, y_1) - \theta(x_0, y_0'), x_1 - x_0 \rangle > 0$$

which again contradicts (12).

Consequently, we conclude that S is a singleton. □

The following characterizes the boundedness of the solution set of $GVIP(F, \theta, K, C)$.

Theorem 3.3.5 *Let K be a closed convex cone in $\mathbf{R}^n$ and C a nonempty subset of $\mathbf{R}^m$. Let F be a point-to-set mapping from K into C and θ a continuous single-valued function from $K \times C$ into $\mathbf{R}^n$. Suppose that the point-to-set mapping $V(x) = \{\theta(x, y) : y \in F(x)\}$ is copositive on K and $V(0) \subset \text{int}(K^*)$. Then the solution set S of $GVIP(F, \theta, K, C)$ is bounded. If, in addition, F is upper continuous and uniformly compact on K, then S is compact.*

Proof. The result is clearly true if S is empty. So let us suppose that S is not empty. If S is unbounded, then there exists a sequence $\{x_n\} \subset S$ such that $||x_n|| \longrightarrow \infty$ as $n \longrightarrow \infty$. For each n, there exists $y_n \in F(x_n)$ such that

$$\langle \theta(x_n, y_n), u - x_n \rangle \geq 0, \forall\ u \in K. \tag{14}$$

Since V is copositive, there exists $z \in F(0)$ such that

$$\langle \theta(x, y) - \theta(0, z), x \rangle \geq 0, \forall\ x \in K, y \in F(x).$$

Thus we have, for each n

$$\langle \theta(x_n, y_n), x_n \rangle \geq \langle \theta(0, z), x_n \rangle. \tag{15}$$

Since $\{x_n\}$ is unbounded, there exists ℓ such that $x_\ell \neq 0$. For this particular ℓ, since $V(0) \subset \mathrm{int}(K^*)$, we have from (15)

$$\langle \theta(x_\ell, y_\ell), x_\ell \rangle > 0.$$

On the other hand, since $0 \in K$, by (14) we have

$$\langle \theta(x_\ell, y_\ell), x_\ell \rangle \leq 0$$

which is a contradiction. Therefore S is bounded. The second assertion follows from Theorem 3.3.1. □

Recall that a point-to-set mapping F is pseudo-monotone on a nonempty set K if for any pair of vectors x_1, x_2 in K and every $y_1 \in F(x_1)$ and every $y_2 \in F(x_2)$, $\langle x_1 - x_2, y_2 \rangle \geq 0$ implies $\langle x_1 - x_2, y_1 \rangle \geq 0$. As pointed out in Karamardian [26], if F is pseudo-monotone and $y_1 \in F(x_1)$, $y_2 \in F(x_2)$, then $\langle x_1 - x_2, y_2 \rangle > 0$ implies $\langle x_1 - x_2, y_1 \rangle > 0$. With this observation, we have the following characterization of compactness of the solution set of $GVIP(F, \theta, K, C)$.

Theorem 3.3.6 *Let K be a closed convex cone in $\mathbf{R}^n$ and C be a nonempty closed convex subset of $\mathbf{R}^m$. Let F be a point-to-set mapping from K into C and $\theta : K \times C \longrightarrow \mathbf{R}^n$ be a continuous single-valued function. Suppose that*

(i) *F is contractible-valued upper continuous and uniformly compact on K,*

(ii) *the point-to-set mapping $V(x) = \{\theta(x, y) : y \in F(x)\}$ is pseudo-monotone on K,*

(iii) *there exist vectors $x_0 \in K$ and $y_0 \in F(x_0)$ such that $\theta(x_0, y_0) \in \mathrm{int}(K^*)$.*

Then the solution set S of $GVIP(F, \theta, K, C)$ is nonempty and compact.

Proof. The fact that S is nonempty and closed follows directly from Theorem 3.2.11 and Theorem 3.3.1. To see that S is also bounded, let

$$D = \{x \in K : \langle \theta(x_0, y_0), x - x_0 \rangle \leq 0\}.$$

Then since $\theta(x_0, y_0) \in \mathrm{int}(K^*)$, D is compact. Now for $x \in K \backslash D$, we have

$$\langle \theta(x_0, y_0), x - x_0 \rangle > 0.$$

Thus by the pseudo-monotonicity of V, we have

$$\langle \theta(x, y), x - x_0 \rangle > 0, \forall\ y \in F(x).$$

Then x can not be a solution to the $GVIP(F, \theta, K, C)$. Consequently, we have $S \subseteq D$ and the result follows. □

The condition (iii) in Theorem 3.3.6 can be looked upon as a Slater-type constraint qualification. We next turn to the question on the convexity of the solution set.

Theorem 3.3.7 *Let K be a nonempty convex subset of $\mathbf{R}^n$ and C a nonempty subset of $\mathbf{R}^m$. Let F be a convex point-to-set mapping from K into C and $\theta : K \times C \longrightarrow \mathbf{R}^n$ be an affine function. Suppose that the point-to-set mapping $V(x) = \{\theta(x,y) : y \in F(x)\}$ is monotone on K. Then the solution set S of $GVIP(F,\theta,K,C)$ is convex.*

Proof. The result is clearly true if S is empty. So let us suppose that S is nonempty. Let $x_1, x_2 \in S$ and $\bar{x} = \alpha x_1 + \beta x_2$ with $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$. Then there exist $y_i \in F(x_i), i = 1,2$ such that for $i = 1,2$

$$\langle \theta(x_i, y_i), u - x_i \rangle \geq 0, \forall\ u \in K. \tag{16}$$

Let $\bar{y} = \alpha y_1 + \beta y_2$. Then $\bar{y} \in F(\bar{x})$ since F is convex. Now for any $u \in K$, we have

$$\begin{aligned}
\langle \theta(\bar{x},\bar{y}), u - \bar{x} \rangle &= \langle \alpha\theta(x_1,y_1) + \beta\theta(x_2,y_2), \alpha(u - x_1) + \beta(u - x_2) \rangle \\
&= \alpha^2 \langle \theta(x_1,y_1), u - x_1 \rangle + \beta^2 \langle \theta(x_2,y_2), u - x_2 \rangle + \\
&\qquad \alpha\beta[\langle \theta(x_1,y_1), u - x_2 \rangle + \theta(x_2,y_2), u - x_1 \rangle] \\
&\geq \alpha\beta[\langle \theta(x_1,y_1), u - x_2 \rangle + \langle \theta(x_2,y_2), u - x_1 \rangle] \qquad \text{(by (16))} \\
&\geq \alpha\beta[\langle \theta(x_1,y_1), x_1 - x_2 \rangle + \theta(x_2,y_2), x_2 - x_1 \rangle] \qquad \text{(by (16))} \\
&= \alpha\beta \langle \theta(x_1,y_1) - \theta(x_2,y_2), x_1 - x_2 \rangle \\
&\geq 0.
\end{aligned}$$

The last inequality follows from the monotonicity of V. Therefore $\bar{x} \in S$. Consequently, S is convex. □

The following corollary is immediate.

Corollary 3.3.8 *Let K be a nonempty convex subset of $\mathbf{R}^n$. Suppose that F is a convex and monotone point-to-set mapping from K into $\mathbf{R}^n$. Then the solution set S of the $GVIP(F,K)$ is convex.*

Remark. A point-to-set mapping F from a nonempty subset K of $\mathbf{R}^n$ into $\mathbf{R}^n$ is said to be *maximal monotone* over K if it is monotone on K and it is not properly contained in any other monotone mapping over K. In [16, Theorem 4.4], Fang and Peterson required F to be maximal monotone over K and deduced the same result as that in Corollary 3.3.8. We note that our conditions on F in Corollary 3.3.8 is different from the maximality condition. To see this, consider the following examples. Let $K = [0,1]$ and $F : K \longrightarrow \mathbf{R}^n$ be defined as $F(x) = \{x\}$ for all $x \in K$. Then it is clear that F is both convex and monotone on K. But it is not maximal monotone over K because F is properly contained in the monotone point-to-set mapping G on K defined as

$$G(x) = \begin{cases} \{x\} & \text{if } 0 \leq x < 1 \\ [1,\infty) & \text{if } x = 1 \end{cases}$$

On the other hand, let $K = \mathbf{R}^2$ and $F(x) = \partial\delta(x|B)$ for all $x \in \mathbf{R}^2$ where B is the closed unit disk

in $\mathbf{R}^2$. That is, $F(x)$ is the subdifferential of the indicator function of B. Then is easy to see that

$$F(x) = \begin{cases} \{0\} & \text{if } x \in \operatorname{int}(B) \\ \{\alpha x : \alpha \geq 0\} & \text{if } x \in \partial(B) \\ \emptyset & \text{else} \end{cases}$$

It is true that F is maximal monotone over $\mathbf{R}^2$ (see, e.g. Rockafellar [38, Corollary 31.5.2, p.340]). But it is clear that F is not convex.

Since any monotone point-to-set mapping is also pseudo-monotone, the following corollary is a direct consequence of Theorem 3.3.6 and Theorem 3.3.7.

Corollary 3.3.9 *Let K be a closed convex cone in $\mathbf{R}^n$ and C be a nonempty closed convex subset of $\mathbf{R}^m$. Let F be a point-to-set mapping from K into C and $\theta : K \times C \longrightarrow \mathbf{R}^n$ be an affine function. Suppose that*

(i) *F is nonempty contractible valued upper continuous and uniformly compact on K,*

(ii) *the point-to-set mapping $V(x) = \{\theta(x, y) : y \in F(x)\}$ is monotone on K,*

(iii) *there exist vectors $x_0 \in K$ and $y_0 \in F(x_0)$ such that $\theta(x_0, y_0) \in \operatorname{int}(K^*)$.*

Then the solution set S of $GVIP(F, \theta, K, C)$ is nonempty, compact and convex. □

4. The Generalized Implicit Complementarity Problem

We begin this section by giving a short introduction on complementarity problems and some possible applications. Let f be a mapping of $\mathbf{R}^n$ into itself. The original *complementarity problem*, denoted by $CP(f)$, is to find a vector $x \in \mathbf{R}^n$ such that

$$x \geq 0, \ f(x) \geq 0, \ \langle x, f(x) \rangle = 0$$

where $x \geq 0$ means all the components of x are nonnegative and $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbf{R}^n$. The task of the above problem is to find a nonnegative x such that its image under f is also nonnegative and perpendicular to itself. When f is nonlinear, $CP(f)$ is called a *nonlinear complementarity problem*. In the case where f is an affine mapping of the form $f : x \mapsto q + Mx$ for some $q \in \mathbf{R}^n$ and $M \in \mathbf{R}^{n \times n}$, the complementarity problem $CP(f)$ is said to be *linear* and is denoted by the pair (q, M). The complementarity problems have many applications, for example, in control and optimization, economics and transportation equilibrium, contact problems in elasticity, fluid flow through porous media, game theory, and mathematical programming. The nonlinear complementarity problem was first introduced and studied by Cottle [6] and Cottle and Dantzig [8] where the notion of *positively bounded Jacobians* was introduced and the proof was constructive in the sense that an algorithm was employed to compute the unique solution. Also see Moré [30, 31, 32].

Given a closed convex cone K of $\mathbf{R}^n$ and a mapping f from $\mathbf{R}^n$ into itself, the *generalized complementarity problem*, denoted by $GCP(f, K)$, is to find a vector $x \in K$ such that

$$f(x) \in K^*, \ \langle x, f(x) \rangle = 0.$$

The idea of generalized complementarity problem was first introduced by Habetler and Price [17] and latter refined by Karamardian [25].

To extend the $GCP(f, K)$, Saigal [41] introduced the following generalized complementarity problem where he considered f to be a point-to-set mapping. Given a closed convex cone K of $\mathbf{R}^n$ and a point-to-set mapping F from K into $\mathbf{R}^n$, the ***generalized complementarity problem***, denoted by $GCP(F, K)$, is to find a vector $x \in K$ and a vector $y \in F(x)$ such that

$$y \in K^*, \ \langle x, y \rangle = 0.$$

It is worth noting that if F is a single-valued function, then the $GCP(F, K)$ reduces to the $GCP(f, K)$.

Motivated by the quasi-variational inequality problem introduced by Mosco [33], Chan and Pang [5] defined a new complementarity problem as follows. Let m and F be respectively, point-to-point and point-to-set mappings of $\mathbf{R}^n$ into itself. Let L be a cone-valued point-to-set mapping on $\mathbf{R}^n$. The ***generalized implicit complementarity problem***, denoted by $GICP(F, m, L)$, is to find a vector $x \in m(x) + L(x)$ and a vector $y \in F(x)$ such that

$$y \in L(x)^*, \ \langle x - m(x), y \rangle = 0.$$

They established some existence results under the assumption that $L(x)$ is a constant closed solid cone for all x.

Recently Parida and Sen [37] introduced the following generalized complementarity problem. Given K a closed convex cone of $\mathbf{R}^n$, C a closed convex subset of $\mathbf{R}^m$, $\theta : K \longrightarrow \mathbf{R}^n$ single-valued function, $F : K \longrightarrow C$ a point-to-set mapping, the ***generalized complementarity problem***, denoted by $GCP(F, \theta, K, C)$, is to find a vector $x \in K$ and a vector $y \in F(x)$ such that

$$\theta(x, y) \in K^*, \ \langle \theta(x, y), x \rangle = 0.$$

It is interesting to observe that Parida and Sen's problem formulation generalizes that of Saigal.

In this section, we consider the following generalized implicit complementarity problem which unifies the above complementarity problems. Let K be a closed convex cone of $\mathbf{R}^n$ and C a nonempty closed convex subset of $\mathbf{R}^m$. Let m be a point-to-point mapping from K into itself and F be a point-to-set mapping of K into C. Let L be a cone valued point-to-set mapping from K into itself and θ a point-to-point mapping from $K \times C$ into $\mathbf{R}^n$. The ***generalized implicit complementarity problem***, denoted by $GICP(F, \theta, m, L, K, C)$ is to find a vector $x \in m(x) + L(x)$ and a vector $y \in F(x)$ such that

$$\theta(x, y) \in L(x)^*, \ \langle \theta(x, y), x - m(x) \rangle = 0.$$

We note that $GICP(F, \theta, m, L, K, C)$ reduces to $GICP(F, m, L)$ if $\theta(x, y) = y$ and $K = C = \mathbf{R}^n$. If $m(x) = 0$ and $L(x) = K$ for all $x \in K$, then $GICP(F, \theta, m, L, K, C)$ reduces to $GCP(F, \theta, K, C)$. Since $GCP(F, \theta, K, C)$ extends $GCP(F, K)$, $GCP(f, K)$ and $CP(f)$, our formulation of the generalized implicit complementarity problem generalizes and unifies the others.

Remark. *Throughout the rest of this section, it is assumed that*

(i) K is a closed convex cone in $\mathbf{R}^n$ and C is a nonempty closed convex subset of $\mathbf{R}^m$.

(ii) *m is a single-valued function from K into itself,*

(iii) *L is a cone-valued point-to-set mapping from K into itself,*

(iv) *F is a point-to-set mapping from K inot C,*

(v) *$X(x) = m(x) + L(x)$ for all $x \in K$,*

(vi) *θ is a continuous single-valued function from $K \times C$ into $\mathbf{R}^n$.*

The relationship between a variational inequality problem and a generalized complementarity problem was first investigated by Karamardian [25]. Karamardian showed that if the set involved in a variational inequality problem is a closed convex cone, then both the variational inequality problem $VI(f, K)$ and the generalized complementarity problem $GCP(f, K)$ have the same solution set. Later Saigal [41], Chan and Pang [5], Noor [35] and Parida and Sen [37] also established the same result on the relationship between the generalized variational inequality problem and the generalized complementarity problem they introduced. Following this direction, we have

Lemma 4.1 *The $GQVIP(X, F, \theta, K, C)$ and the $GICP(F, \theta, m, L, K, C)$ have the same solution set.*

Proof. Let x be a vector in K that solves $GQVIP(X, F, \theta, K, C)$. Then $x \in X(x)$, and there exists $y \in F(x)$ such that

$$\langle \theta(x, y), z - x \rangle \geq 0, \forall\ z \in X(x).$$

Since $x - m(x) \in L(x)$ and $L(x)$ is a cone, we have $2(x - m(x)) \in L(X)$. Thus $2x - m(x) \in X(x)$, and we have $\langle \theta(x, y), x - m(x) \rangle \geq 0$. On the other hand, since $m(x) \in X(x)$, we have $\langle \theta(x, y), m(x) - x \rangle \leq 0$. Consequently, $\langle \theta(x, y), x - m(x) \rangle = 0$. Now for each $z \in L(x)$, we have $m(x) + z \in X(x)$. Accordingly,

$$\langle \theta(x, y), z \rangle = \langle \theta(x, y), m(x) + z - x \rangle \geq 0.$$

Therefore $\theta(x, y) \in L(x)^*$. Hence (x, y) solves the $GICP(F, \theta, m, L, K, C)$. Conversely, suppose x solves the $GICP(F, \theta, m, L, K, C)$. Then $x \in m(x) + L(x)$, and there exists $y \in F(x)$ such that

$$\theta(x, y) \in L(x)^*, \ \langle \theta(x, y), x - m(x) \rangle = 0.$$

For any $z \in m(x) + L(x)$, there exists $v \in L(x)$ such that $z = m(x) + v$. Thus we have

$$\langle \theta(x, y), z - x \rangle = \langle \theta(x, y), v \rangle \geq 0.$$

Therefore (x, y) solves $GQVIP(X, F, \theta, K, C)$. □

Corollary 4.2

(i) *$GCP(F, \theta, K, C)$ and $GVIP(F, \theta, K, C)$ have the same solution set.*

(ii) *$GCP(F, K)$ and $GVIP(F, K)$ have the same solution set.* □

The following gives an existence result for the GCP.

Lemma 4.3 *Let $B \subset K$ with $\text{int}_K(B) \neq \emptyset$. If x solves $GVIP(F, \theta, B, C)$ and $x \in \text{int}_K(B)$, then x solves $GCP(F, \theta, K, C)$.*

Proof. Since x solves $GVIP(F, \theta, B, C)$, there exists $y \in F(x)$ such that

$$\langle \theta(x, y), u - x \rangle \geq 0, \forall\, u \in B.$$

For each $z \in K$, since $x \in \text{int}_K(B)$, there exists $0 < \lambda < 1$ such that $\lambda z + (1 - \lambda)x \in B$. Then

$$\langle \theta(x, y), \lambda z + (1 - \lambda)x - x \rangle = \lambda \langle \theta(x, y), z - x \rangle \geq 0.$$

Thus $\langle \theta(x, y), z - x \rangle \geq 0$. Hence x solves $GVIP(F, \theta, K, C)$ and the result follows from Corollary 4.2 (i). □

With the aid of Lemma 4.1 and the existence results for $GQVIP$ in Section 3, we obtain the following existence results for the $GICP(F, \theta, m, L, K, C)$.

Theorem 4.4 *Suppose that there exists an $r > 0$ such that the following conditions hold:*

(i) *F is contractible-valued upper continuous and uniformly compact on B_r,*

(ii) *$Y(x) = X(x) \cap B_r$ is a nonempty convex-valued continuous point-to-set mapping on B_r,*

(iii) *for each $x \in X(x) \cap C_r$, there is a $u \in X(x) \cap \text{int}_K(B_r)$ such that*

$$\max\nolimits_{y \in F(x)} \langle \theta(x, y), u - x \rangle \leq 0.$$

Then there exists a solution to the $GICP(F, \theta, m, L, K, C)$.

Proof. This follows directly from Corollary 3.1.9 and Lemma 4.1. □

Theorem 4.5 *Suppose that there exists a compact subset B of K such that the following conditions hold:*

(iii) *F is contractible-valued upper continuous and uniformly compact on B,*

(ii) *$Y(x) = X(x) \cap B$ is a nonempty convex-valued continuous point-to-set mapping on B,*

(iii) *for each $x \in B$ and for each $z \in X(x) \backslash B$, there exists a vector $u \in Y(x)$ such that*

$$\max\nolimits_{y \in F(x)} \langle \theta(x, y), u - z \rangle \leq 0.$$

Then there exists a solution to the $GICP(F, \theta, m, L, K, C)$.

Proof. This follows directly from Theorem 3.1.10 and Lemma 4.1. □

Theorem 4.6 *Suppose that*

(i) *F is contractible-valued upper continuous and uniformly compact on K,*

(ii) *there exists a vector $x_0 \in \bigcap_{x\in K} X(x)$ such that*

$$\lim_{\|x\|\longrightarrow\infty,\ x\in X(x)} \max_{y\in F(x)} \langle \theta(x,y), x_0 - x\rangle < 0,$$

(iii) *there exists a $\rho_0 > 0$ such that $X(x) \cap B_\rho$ is a nonempty convex-valued continuous point-to-set mapping for all $\rho \geq \rho_0$.*

Then there exists a solution to the $GICP(F, \theta, m, L, K, C)$. □

The following corollary is immediate.

Corollary 4.7 *Let $F : K \longrightarrow \mathbf{R}^n$ be a point-to-set mapping which is upper continuous, uniformly compact and contractible-valued on K. Suppose there exists $z \in K$ such that*

$$\lim_{\|x\|\longrightarrow\infty,\ x\in K} (\inf_{y\in F(x)} \langle\ x - z, y\ \rangle) = \infty.$$

Then $GCP(F, K)$ has a solution. □

The following theorem is a direct consequence of Theorem 3.2.4 and Lemma 4.1.

Theorem 4.8 *Suppose that there exists an $r > 0$ and $x_0 \in K$ such that*

(i) *$x_0 \in (\bigcap_{x\in C_r} X(x)) \cap \operatorname{int}_K(B_r)$ and*

$$\inf_{x\in X(x)\cap C_r} \inf_{y\in F(x)} \langle \theta(x,y), x - x_0\rangle \geq 0,$$

(ii) *F is contractible-valued upper continuous and uniformly compact on B_r,*

(iii) *$Y(x) = X(x) \cap B_r$ is a nonempty convex-valued continuous point-to-set mapping on B_r.*

Then there exists a solution to the $GICP(F, \theta, m, L, K, C)$.

The following gives an existence result for the *GICP* under the strong copositivity condition.

Theorem 4.9 *Suppose that*

(i) *there exists $x_0 \in \bigcap_{x\in K} X(x)$ such that the point to set mapping $V(x) = \{\theta(x,y) : y \in F(x)\}$ is strongly copositive with respect to X at x_0 on K,*

(ii) *F is contractible-valued upper continuous and uniformly compact on K,*

(iii) *there exists a $\rho_0 > 0$ such that $X(x) \cap B_\rho$ is a nonempty convex-valued continuous point-to-set mapping for all $\rho \geq \rho_0$.*

Then there exists a solution to the $GICP(F, \theta, m, L, K, C)$.

Proof. This follows directly from Theorem 3.2.6 and Lemma 4.1. □

We note that the remark following Theorem 3.2.6 is also valid for Theorem 4.9. The following corollary is immediate.

Corollary 4.10 *Suppose that*

(i) *there exists $x_0 \in \bigcap_{x \in K} X(x)$ such that the point to set mapping $V(x) = \{\theta(x, y) : y \in F(x)\}$ is strongly monotone with respect to X on K,*

(ii) *F is contractible-valued upper continuous and uniformly compact on K,*

(iii) *there exists a $\rho_0 > 0$ such that $X(x) \cap B_\rho$ is a nonempty convex-valued continuous point-to-set mapping for all $\rho \geq \rho_0$.*

Then there exists a solution to the $GICP(F, \theta, m, L, K, C)$. □

The following gives an existence result for the $GICP$ under the pseudo-monotonicity condition.

Theorem 4.11 *Suppose that*

(i) *F is contractible-valued upper continuous and uniformly compact on K,*

(ii) *the point to set mapping $V(x) = \{\theta(x, y) : y \in F(x)\}$ is monotone or pseudo-monotone with respect to X on K,*

(iii) *there exist vectors $x_0 \in \bigcap_{x \in K} X(x)$ and $y_0 \in F(x_0)$ such that*

$$\theta(x_0, y_0) \in \operatorname{int}(\bigcup_{x \in K} X(x))^*,$$

(iv) *there exists a $\rho_0 > 0$ such that $X(x) \cap B_\rho$ is a nonempty convex-valued continuous point-to-set mapping for all $\rho \geq \rho_0$.*

Then there exists a solution to the $GICP(F, \theta, m, L, K, C)$.

Proof. This follows directly from Theorem 3.2.11 and Lemma 4.1. □

In the case where the point-to-set mapping L is constant, the condition that $X(x) \cap B_\rho$ is continuous on B_ρ for large ρ is automatically true if the function m is continuous. We then have

Theorem 4.12 *Suppose that*

(i) *F is contractible-valued upper continuous and uniformly compact on K,*

(ii) *there exists a vector $x_0 \in \bigcap_{x \in K} X(x)$ such that*

$$\lim_{\|x\| \longrightarrow \infty, x \in X(x)} \max_{y \in F(x)} \langle \theta(x, y), x_0 - x \rangle < 0,$$

(iii) *there exists a vector* $u_0 \in K$ *such that* $u_0 - m(x) \in L, \forall\, x \in K$.

Then there exists a solution to the $GICP(F, 0, m, L, K, C)$. $\square$

Theorem 4.12 can be proved by a standard argument as in [5, Theorem 4.2]. We close this chapter by remarking that most of the existence results for $GICP$ rely very heavily on the existence results for $GQVIP$. This suggests that we should exploit other approach rather than $GQVIP$.

5. Applications

In this section we shall give several applications of our general problems. Mathematical programming and equilibrium programming are the two major areas of the applications. The applications are: minimization problems involving "invex" functions, generalized dual problems and saddle point problems, equilibrium problems involving markets with utility, equilibrium problems involving abstract economies, generalized Nash equilibrium problems, and quasi-variational inequality problems of obstacle type. In all these applications, we require relatively weak conditions to ensure the existence of solutions to the problems under consideration.

5.1. Minimization Problems Involving "Invex" Functions

Recently Hanson [18] introduced into optimization theory a broad generalization of convexity for differentiable functions on $\mathbf{R}^n$ which was called invex by Craven [10]. Let K be a nonempty subset of $\mathbf{R}^n$. A differentiable function f on K is *invex* if there exists a vector function τ from $K \times K$ into $\mathbf{R}^n$ such that

$$f(x) - f(y) \geq \langle \nabla f(y), \tau(x, y) \rangle, \forall\, x, y \in K.$$

Hanson showed that both weak duality and Kuhn-Tucker sufficient results, in constrained optimization, hold with the invex conditions.

We note that if $\tau(x, y) = x - y$, then the invexity condition for f reduces to convexity condition. An example of invex function is the following (Hanson [18]). Let $K = \{(x, y) \in \mathbf{R}^2_+ : x^2 + y^2 \leq 3/2\}$ and let function f be defined as $f(x, y) = x - \sin y$ for all $(x, y) \in K$. Then f is invex with respect to τ, where

$$\tau((\alpha, \beta), (\gamma, \delta)) = ((\sin\alpha - \sin\gamma)/\cos\gamma, (\sin\beta - \sin\delta)/\cos\delta).$$

For more details on the concept of invexity, see Rueda and Hanson [40], Craven [10] and Jeyakumar [24].

Consider the following minimization problem:

$$\min_{x \in K} f(x) \tag{17}$$

where K is a nonempty subset of $\mathbf{R}^n$ and f is a differentiable invex function with respect to τ on K.

We associate with problem (17) the following variational inequality problem: find $x \in K$ such that

$$\langle \nabla f(x), \tau(u, x) \rangle \geq 0, \forall\, u \in K. \tag{18}$$

It is easy to see that if x is a solution to problem (18), then x is a solution to problem (17). Consequently, we have

Theorem 5.1.1 *Let K be a nonempty convex subset of $\mathbf{R}^n$ and let f be a continuously differentiable invex function with respect to a continuous function τ on K. Suppose that*

(i) $\tau(x, x) = 0$ *for all* $x \in K$,

(ii) *for each fixed* $x \in K$, $\langle \nabla f(x), \tau(u, x)\rangle$ *is convex in* $u \in K$,

(iii) *there exists a vector* $x_0 \in K$ *such that*

$$\lim_{\|x\| \longrightarrow \infty,\ x \in K} \langle \nabla f(x), \tau(x_0, x)\rangle < 0.$$

Then there exists a solution to problem (17).

Proof. Let X be a constant point-to-set mapping from K into itself with $X(x) = K$ for all $x \in K$. Let $\theta : K \times K \longrightarrow \mathbf{R}^n$ be defined as $\theta(x, y) = \nabla f(x)$. Then problem (17) is equivalent to $GQVIP(X, X, \theta, \tau, K, K)$. By Theorem 3.2.1, the latter problem has a solution. Hence there exists a solution to problem (17). □

Remarks.

(i) The function τ in the above example satisfies $\tau(x, x) = 0$ for all $x \in K$. Hence the condition (i) of Theorem 6.1.1 is not restrictive.

(ii) There are some other conditions on f that will ensure the existence of solution to problem (17). For instance, if the condition (iii) of Theorem 5.1.1 is replaced by the condition that ∇f is τ-monotone on K, then the corresponding $GQVIP(X, X, \theta, \tau, K, K)$ has a solution by Theorem 3.2.10. Consequently, there is a solution to problem (17).

5.2. Generalized Dual Problems and Saddle Point Problems

Our second application is to generalized dual problems and saddle point problems. A basic result in optimization theory is that under some conditions, a saddle point of the Lagrangian function is equivalent to an optimum of the associated convex programming problem satisfying a constraint qualification. This result has been significantly demonstrated in economic literature (see,e.g., Heal [21]). This is the impetus of our application in this section. First, let us introduce the formulation of the problems. Let K and C be nonempty subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. Let φ be a real function on $K \times C$. Let X and F be nonempty valued point-to-set mappings from K into K and C, respectively.

(5.2.1) Generalized Problem I (GPI): Find $(\bar{x}, \bar{y}) \in S$ such that

$$\varphi(\bar{x}, \bar{y}) = \inf_{(x,y) \in S} \varphi(x, y)$$

where

$$S = \{(x, y) : x \in X(x), y \in F(x), \varphi(x, y) = \sup_{u \in F(x)} \varphi(x, u)\}.$$

(5.2.2) Generalized Problem II (GPII): Find $(\bar{x}, \bar{y}) \in T$ such that

$$\varphi(\bar{x}, \bar{y}) = \sup_{(x,y)\in T} \varphi(x, y)$$

where

$$T = \{(x, y) : x \in X(x), y \in F(x), \varphi(x, y) = \inf_{v\in X(x)} \varphi(v, y)\}.$$

(5.2.3) Generalized Saddle Point Problem (GSPP): Find $\bar{x} \in X(\bar{x})$ and $\bar{y} \in F(\bar{x})$ such that

$$\varphi(\bar{x}, y) \leq \varphi(\bar{x}, \bar{y}) \leq \varphi(x, \bar{y})$$

for all $x \in X(\bar{x})$ and all $y \in F(\bar{x})$.

We note that the above problems may not have any solution at all. Also if we let $X(x) = K$ and $F(x) = C$ for all $x \in K$, then the definitions of (GSPP), (GPI) and (GPII) reduce to the definitions of (SPP), (PI) and (PII) introduced by Mangasarian and Ponstein [27].

The following lemma points out the relationship between these problems.

Lemma 5.2.4 *Let K and C be nonempty subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. Let X and F be nonempty valued point-to-set mappings from K into K and C, respectively. Let φ be a real function on $K \times C$. If $(\bar{x}, \bar{y})$ is a solution of the (GSPP), then $(\bar{x}, \bar{y})$ is also a solution of the (GPI) and (GPII), and conversely.*

Proof. Assume $(\bar{x}, \bar{y})$ is a solution of (GSPP). Then clearly $(\bar{x}, \bar{y}) \in S \cap T$. Let $(x, y) \in S$. Then we have $\varphi(x, y) \geq \varphi(x, \bar{y})$. But $\varphi(x, \bar{y}) \geq \varphi(\bar{x}, \bar{y})$ since $(\bar{x}, \bar{y}) \in T$. Thus $\varphi(x, y) \geq \varphi(\bar{x}, \bar{y})$. Hence $(\bar{x}, \bar{y})$ is a solution of (GPI). Similarly, $(\bar{x}, \bar{y})$ is a solution of (GPII). The converse is clear since if $(\bar{x}, \bar{y})$ is a solution of both (GPI) and (GPII), then

$$\varphi(\bar{x}, \bar{y}) = \sup_{y\in F(\bar{x})}\varphi(\bar{x}, y) = \inf_{x\in X(\bar{x})}\varphi(x, \bar{y}). \ \square$$

We now associate with (GSPP) the following generalized quasi-variational inequality problem: Find $\bar{x} \in X(\bar{x})$ and $\bar{y} \in G(\bar{x})$ such that

$$\langle \nabla_x\varphi(\bar{x}, \bar{y}), \tau(x, \bar{x})\rangle \geq 0, \forall\ x \in X(\bar{x}) \tag{19}$$

where

$$G(x) = \{y \in F(x) : \varphi(x, y) = \sup_{u\in F(x)}\varphi(x, u)\}.$$

The following lemma establishes the relationship between (GSPP) and problem (19).

Lemma 5.2.5 *Suppose $\varphi(x, y)$ is invex in $x \in K$ with respect to τ for each fixed $y \in C$. If $(\bar{x}, \bar{y})$ is a solution of (19), then $(\bar{x}, \bar{y})$ is a solution of (GSPP).*

Proof. Assume $(\bar{x}, \bar{y})$ solves (19). By the invexity of φ, we have for any $x \in X(\bar{x})$

$$\varphi(x, \bar{y}) - \varphi(\bar{x}, \bar{y}) \geq \langle\ \nabla_x\varphi(\bar{x}, \bar{y}), \tau(x, \bar{x})\ \rangle \geq 0.$$

Therefore $\varphi(x, y) \geq \varphi(\bar{x}, \bar{y})$ for all $x \in X(\bar{x})$. On the other hand, since $\bar{y} \in G(\bar{x})$, we have for all $y \in F(\bar{x})$

$$\varphi(\bar{x}, y) \leq \varphi(\bar{x}, \bar{y}).$$

Hence $(\bar{x}, \bar{y})$ solves (GSPP). □

Thus the question of existence of solution to (GPI), (GPII) and (GSPP) can be investigated via (19). Consequently, we have the following existence result for (GSPP).

Theorem 5.2.6 *Let K and C be nonempty subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. Let X and F be nonempty valued point-to-set mappings from K into K and C, respectively, and τ be a continuous function from $K \times K$ into $\mathbf{R}^n$. Suppose that*

(i) $\tau(x, x) = 0, \forall\ x \in K$,

(ii) *for each fixed $(x, y) \in K \times C$, $\langle \nabla_x \varphi(x, y), \tau(u, x) \rangle$ is convex in $u \in X(x)$,*

(iii) *F is convex valued, continuous and uniformly compact on K,*

(iv) *$\varphi(x, y)$ is invex with respect to τ on K for each fixed $y \in C$, and concave in $y \in C$ for each fixed $x \in K$.*

(v) *there exists a vector $x_0 \in \bigcap_{x \in K} X(x)$ such that*

$$\lim_{\|x\| \longrightarrow \infty,\ x \in X(x)} \max_{y \in F(x)} \langle \nabla_x \varphi(x, y), \tau(x_0, x) \rangle < 0,$$

(vi) *there exists a $\rho_0 > 0$ such that $X(x) \cap B_\rho$ is a nonempty convex valued continuous point-to-set mapping for all $\rho \geq \rho_0$.*

Then there exists a solution to the (GSPP).

Proof. For each $x \in K$, let

$$G(x) = \{y \in F(x) : \varphi(x, y) = \sup_{u \in F(x)} \varphi(x, u)\}.$$

It is easy to see that $G(x)$ is upper continuous and uniformly compact. Since $\varphi(x, y)$ is concave in $y \in C$, by (ii) $G(x)$ is compact and convex for all $x \in K$. All the conditions of Theorem 3.2.1 are satisfied. Therefore, there exists a solution to (19) by Theorem 3.2.1. Hence there exists a solution to (GSPP) by Lemma 5.2.5. □

The following corollary is immediate.

Corollary 5.2.7 *Under the conditions of Theorem 5.2.6, there exists a solution to (GPI) and (GPII).* □

Remarks.

(i) It is worth noting that the Generalized Saddle Point Problem can not be approached by any other variational inequality problem that has been introduced.

(ii) The other existence results in Section 3.2 can also be employed to ensure the existence of solutions of (GSPP) and (GPI) and (GPII).

(iii) Recently, Jeyakumar [24] has extended the saddle point theorems to hold for a more general class of functions, called the ρ-invex functions which is an extension of the class of invex functions.

5.3. Equilibrium Problems Involving Markets with Utility

In this section, we shall apply Theorem 3.1.2 to obtain an existence result for equilibrium of a market with utility. Let us first introduce the notion of a market with utility.

Let $I = \{1, \ldots, m\}$ and for each $i \in I$, let $X_i \subseteq \mathbf{R}^n$ be a closed set that is bounded from below. Let a_i be a specified element of X_i. We call I the set of *agents*, X_i the *commodity set* of ith agent and a_i the *initial allocation* of the ith agent. For each i, there is a continuous function u_i from X_i into $\mathbf{R}$ which is the *utility function* of the ith agent. Let $X = (X_i)_{i\in I}$, $U = (u_i)_{i\in I}$, $A = (a_i)_{i\in I}$. Then the 4-tuple (I, X, U, A) is said to be a *market with utility*. Let

$$V = \{x : x = (x_i)_{i\in I}, x_i \in X_i, \forall\, i, \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} a_i\}$$

be the *allocation set* and

$$B_p^i = \{x_i \in X_i : \langle p, x_i\rangle \leq \langle p, a_i\rangle\}$$

be the *budget set* for the ith agent where $p \in P^n = \{p \in \mathbf{R}_+^n : \sum_{j=1}^{n} p_j = 1\}$, the *price set*. A point $(p^*, x^*) \in P^n \times V$ is said to be an *equilibrium* for a market with utility (I, X, U, A) if, for $i = 1, \ldots, m$

$$u_i(x_i^*) = \max\{u_i(x_i) : x_i \in B_{p^*}^i\}.$$

Intuitively, an equilibrium is characterized by the property that given a price vector, there is a reallocation of goods, such that every agent maximizes his utility function within the limit of his budget. We have the following existence result for the equilibrium of a market with utility.

Theorem 5.3.1 *Let (I, X, U, A) be a market with utility. Suppose that*

(i) *X_i is convex and there exists $\bar{x}_i \in X_i$ with $\bar{x}_i < a_i$, for all i,*

(ii) *$\sum_{i=1}^{m} u_i(x_i)$ is quasiconcave in $x = (x_i)_{i\in I} \in (\mathbf{R}^n)^m$.*

Then there exists an equilibrium to the market with utility (I, X, U, A).

Proof. Let Y be a point-to-set mapping from $P^n \times V$ into itself be defined as $Y(p, x) = \{p\} \times \prod_{i=1}^{m} B_p^i$. Let F be a constant point-to-set mapping from $P^n \times V$ into itself. Clearly, Y is nonempty convex valued and upper continuous. Furthermore, by Lemma 1.6 of [39, Chapter 5], $B^i : P^m \longrightarrow X_i$ is lower continuous for each i under the condition (i). Therefore Y is continuous. Next, let θ and τ single-valued functions from $(P^n \times V) \times (P^n \times V)$ into $\mathbf{R}^m$ be defined as $\theta((p, x), (q, y)) = e$, where

e is the unity vector in $\mathbf{R}^m$ and $\tau((p,x),(q,y)) = (u_i(y_i) - u_i(x_i))_{i\in I}$. Then all the conditions of Theorem 3.1.2 are satisfied. Thus by Theorem 3.1.2, there exists $(p^*, x^*) \in Y(p^*, x^*)$ such that

$$\sum_{i=1}^{m}(u_i(x_i^*) - u_i(x_i)) \geq 0, \ \forall\ (x_i) \in \prod_{i=1}^{m} B_{p^*}^i.$$

It is easy to see that (p^*, x^*) is an equilibrium for the market with utility (I, X, U, A). □

Remark. In Theorem 1.11 of [39, Chapter 5], it is assumed that u_i is monotone and concave for each i. Therefore it can be seen that the condition (ii) of Theorem 5.3.1 is weaker than that in Theorem 1.11 of [39, Chapter 5].

5.4. Equilibrium Problems Involving Abstract Economies

The notion of abstract economies, which is a generalization of Nash equilibrium problems, was introduced by Debreu [11]. In a Nash equilibrium problem the strategy choices of agents are made independently, whereas, in an abstract economy, the set of strategies available to each agent depends on strategy choices of the other agents. To be more precise, we recall the definition of an equilibrium of an abstract economy. Suppose there are m agents characterized by a subscript $i = 1, \ldots, m$. The ith agent chooses an action x_i from his strategy set $V_i \subseteq \mathbf{R}^{n_i}$. Let $V = \prod_{i=1}^m V_i \subseteq \mathbf{R}^n$ with $n = \sum_{i=1}^m n_i$. The payoff to the ith agent is a function $f_i(x)$ from V into the completed real line.

Let $\bar{x}_i$ be the $(m-1)$-tuple $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$ and similarly let $\bar{V}_i$ be the product $V_1 \times \cdots \times V_{i-1} \times V_{i+1} \times \cdots \times V_m$. We can interpret $\bar{x}_i$ the actions of all the others. Given $\bar{x}_i$, the choice of the ith agent is restricted to a nonempty set $A_i(\bar{x}_i) \subseteq V_i$. The ith agent chooses $x_i \in A_i(\bar{x}_i)$ so as to maximize $f_i(\bar{x}_i, x_i)$. The $3m$-tuple $[V_i, f_i, A_i(\bar{x}_i)]_{i=1}^m$ is said to be an *abstract economy*. The point x^* is said to be an *equilibrium* of an abstract economy $[V_i, f_i, A_i(\bar{x}_i)]_{i=1}^m$, if for all $i = 1, \ldots, m$,

$$x_i^* \in A_i(\bar{x}_i^*) \text{ and } f_i(x^*) = \max_{x_i \in A_i(\bar{x}_i^*)} f_i(\bar{x}_i^*, x_i).$$

Thus an equilibrium point is characterized by the property that given the actions of the other agents, each agent is maximizing his own payoff function over the set of his feasible actions in view of the other agents' actions.

We now associate with the equilibrium problem of an abstract economy the following generalized quasi-variational inequality problem: Find $x^* \in X(x^*)$ such that

$$\langle e, \tau(x,y)\rangle \geq 0, \ \forall\ x \in X(x^*) \tag{20}$$

where $X(x) = \prod_{i=1}^m A_i(\bar{x}_i)$ for all $x \in V$, $\tau(x,y) = (f_i(y) - f_i(\bar{y}_i, x_i))_{i=1}^m$ and e is the unity vector in $\mathbf{R}^m$.

It is easy to see that x^* is an equilibrium of an abstract economy $[V_i, f_i, A_i(\bar{x}_i)]_{i=1}^m$ if and only if x^* is a solution to (20). We then have the following existence result for the equilibrium of an abstract economy.

Theorem 5.4.1 *Given an abstract economy $[V_i, f_i, A_i(\bar{x}_i)]_{i=1}^m$ which satisfies the following conditions: for each $i = 1, \ldots, m$*

(i) V_i is nonempty compact and convex,

(ii) $A_i(\bar{x}_i)$ *is a convex-valued continuous point-to-set mapping on* $\bar{V}_i$,

(iii) f_i *is continuous,*

(iv) *for each* $x \in V$, $\langle e, \tau(u,x)\rangle$ *is quasiconvex in* $u \in X(x)$ *where* τ *and* X *are as in (20).*

Then there exists an equilibrium point for $[V_i, f_i, A_i(\bar{x}_i)]_{i=1}^m$.

Proof. Let $C = K = V$ and F a constant point-to-set mapping on K. Let $\theta, \tau_1 : K \times K \longrightarrow \mathbf{R}^n$ be defined as $\theta(x,y) = (e,0)$ and $\tau_1(x,y) = (\tau(x,y),0)$ respectively, where $n = \sum_{i=1}^m n_i$ and 0 is understood to be a zero vector in $\mathbf{R}^{n-m}$. Then the equilibrium problem involving the abstract economy $[V_i, f_i, A_i(\bar{x}_i)]_{i=1}^m$ is equivalent to $GQVIP(X, F, \theta, \tau_1, V, V)$. By Theorem 3.1.2, the latter problem has a solution. Hence the result follows. □

Remarks.

(i) Condition (ii) of Theorem 5.4.1 is equivalent to the condition that the function $\sum_{i=1}^m f_i(\bar{x}_i, u_i)$ is quasiconcave in $u \in X(x)$.

(ii) Our definition of an abstract economy is slightly different from the one in [5] where f_i is defined on $V \times V_i$. Also in [5], Chan and Pang did not use the approach of variational inequality problem to obtain the result of Theorem 5.4.1.

For the case that V_i is not necessarily compact, we have the following existence result.

Theorem 5.4.2 *Given an abstract economy* $[V_i, f_i, A_i(\bar{x}_i)]_{i=1}^m$ *which satisfies the following conditions: for each* $i = 1, \ldots, m$

(i) f_i *is continuous,*

(ii) *for each* $x \in V$, $\langle e, \tau(u,x)\rangle$ *is convex in* $u \in X(x)$ *where* τ *and* X *are as in (38),*

(iii) *there exists a vector* $x^0 \in \bigcap_{x \in V} X(x)$ *such that*

$$\lim_{\|x\| \longrightarrow \infty,\ x \in X(x)} \left(\sum_{i=1}^m f_i(\bar{x}_i^0, x_i)\right) > 0,$$

(iv) *there exists a* $\rho_0 > 0$ *such that* $X(x) \cap B_\rho$ *is a nonempty convex valued continuous point-to-set mapping for all* $\rho \geq \rho_0$.

Then there exists an equilibrium point.

Proof. This follows from Theorem 3.2.1 and the note after (20). □

For more details on abstract economy, we refer interested readers to Debreu [11].

5.5. Generalized Nash Equilibrium Problems

The concept of Nash equilibrium (Nash [34]) was extended by Ichiishi [23] to include additional joint constraints on agents' actions which cut across all agents simultaneously. The formal definition of generalized Nash equilibrium is as follows.

Suppose there are m *agents* in a noncooperative game characterized by a subscript $i = 1, \ldots, m$. The ith agent is represented by a *strategy vector* $x_i \in V_i \subseteq \mathbf{R}^{n_i}$ (n_i being a positive integer), a point-to-set mapping $X_i : V \longrightarrow V_i$, and a *utility function* $u_i : V \longrightarrow \mathbf{R}$ where $V = \prod_{i=1}^m V_i \subseteq \mathbf{R}^n$ with $n = \sum_{i=1}^m n_i, X = \prod_{i=1}^m X_i$, and $U = (u_1, \ldots, u_m)$. A *generalized Nash equilibrium* $x^* \in V$ of the game $GNE(V, X, U)$ is defined as a point at which no agent can unilaterally increase his utility function given the constraints imposed on him by the other agents:

$$u_i(x^*) \geq u_i(x_i, \bar{x}_i^*),\ \forall\ x_i \in X_i(x^*)$$

where $\bar{x}_i$ is the $(m-1)$-tuple $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$.

We note that if $X_i(x) = V_i$ for all i and x, then the above definition of generalized Nash equilibrium reduces to the definition of Nash equilibrium. Let $e \in \mathbf{R}^m$ be the unity vector and $\tau : V \times V \longrightarrow \mathbf{R}^m$ be defined as $\tau(x, y) = (u_i(y) - u_i(x_i, \bar{y}_i))_{i=1}^m$. We associate with the generalized Nash equilibrium problem the following generalized quasi-variational inequality problem: Find $x^* \in X(x^*)$ such that

$$\langle e, \tau(x, y) \rangle \geq 0,\ \forall\ x \in X(x^*). \tag{21}$$

It is easy to see that x^* is a generalized Nash equilibirum of the game $GNE(V, X, U)$ if and only if x^* is a solution of (21). We then have the following existence result for the generalized Nash equilibrium problem. The proof is exactly the same as that in Theorem 5.4.1.

Theorem 5.5.1 *Given a generalized m-person noncooperative game (V, X, U) which satisfies the following conditions: for each $i = 1, \ldots, m$*

(i) *V_i is nonempty compact and convex,*

(ii) *$X_i(x)$ is a convex valued continuous point-to-set mapping on V,*

(iii) *u_i is continuous,*

(iv) *for each $x \in V$, $\langle e, \tau(u, x) \rangle$ is quasiconvex in $u \in X(x)$ where τ is as in (21).*

Then there exists a generalized Nash equilibrium for (V, X, U). □

For the case that V_i is not necessarily compact, we have the following existence result. The proof is the same as that in Theorem 5.4.2.

Theorem 5.5.2 *Given a generalized m-person noncooperative game (V, X, U) which satisfies the following conditions: for each $i = 1, \ldots, m$*

(i) *u_i is continuous,*

(ii) *for each $x \in V$, $\langle e, \tau(u, x) \rangle$ is convex in $u \in X(x)$ where τ and X are as in (21),*

(iii) *there exists a vector* $x^0 \in \bigcap_{x \in V} X(x)$ *such that*

$$\lim_{\|x\| \longrightarrow \infty,\ x \in X(x)} \left(\sum_{i=1}^{m} (u_i(x) - u_i(x_i^0, \bar{x}_i))\right) < 0,$$

(iv) *there exists a* $\rho_0 > 0$ *such that* $X(x) \cap B_\rho$ *is a nonempty convex-valued continuous point-to-set mapping for all* $\rho \geq \rho_0$.

Then there exists a generalized Nash equilibrium for the game (V, X, U).

Remark. There is no differentiability requirement on the utility function u_i for all i in Theorem 5.5.1 and 5.5.2. The reason is that we deal with each agent's utility maximizing problem directly without using the first-order optimality conditions.

We refer interested readers to a survey paper by Harker and Pang [19] where there is a thorough discussion on the Nash equilibrium and generalized Nash equilibrium problems.

5.6. Quasi-Variational Inequality Problems of Obstacle Type

In this section we shall be concerned with the quasi-variational inequality problems of obstacle type formulated as follows. Let K be a closed convex cone in $\mathbf{R}^n$ and $\leq_K$ the partial order induced by K, that is, $x \leq_K y$ if and only if $x - y \in K$ for all $x, y \in \mathbf{R}^n$. Let f, m be functions from $\mathbf{R}^n$ into itself. The *quasi-variational inequality problem of obstacle type* is to find $x^* \in \mathbf{R}^n$ such that

$$x^* \leq_K m(x^*),\ \langle f(x), x - x^* \rangle \geq 0,\ \forall\ x \leq_K m(x^*). \tag{22}$$

It is interesting to note that if $K = \mathbf{R}^n_+$ and $m(x) = 0$ for all $x \in \mathbf{R}^n$, then problem (22) is equivalent to a nonlinear complementarity problem. We now associate with problem (22) the following generalized implicit complementarity problem. Let X be a point-to-set mapping from $\mathbf{R}^n$ into itself defined as $X(x) = m(x) + K$ for all $x \in \mathbf{R}^n$. Find $x^* \in m(x^*) + K$ such that

$$f(x^*) \in K^*, \langle f(x^*), x^* - m(x^*) \rangle = 0. \tag{23}$$

It is easy to see that problem (22) is equivalent to problem (23) by Lemma 4.1. We have the following existence result for problem (22).

Theorem 5.6.1 *Let* K *be a closed solid convex cone in* $\mathbf{R}^n$. *Let* f *and* m *be continuous functions from* $\mathbf{R}^n$ *into itself and* $X(x) = m(x) + K$ *be a point-to-set mapping from* $\mathbf{R}^n$ *into itself. Suppose that*

(i) *there exists a vector* $x_0 \in \bigcap_{x \in \mathbf{R}^n} X(x)$ *such that*

$$\lim_{\|x\| \longrightarrow \infty,\ x \in X(x)} \langle f(x), x_0 - x \rangle < 0,$$

(ii) *there exists a vector* $u_0 \in \mathbf{R}^n$ *such that* $u_0 - m(x) \in K, \forall\ x \in \mathbf{R}^n$.

Then there exists a solution to problem (22).

Proof. This result follows from Theorem 3.12 and the note above. □

We note that the condition (i) of Theorem 5.6.1 can be replaced by the condition that f is strongly copositive or strongly monotone on $\mathbf{R}^n$. For the quasi-variational inequality problems of obstacle type in a reflexive Banach lattice, we refer readers to the paper by Dolcetta and Matzeu [13] and the references therein.

There are other areas of applications of the theory of complementarity problems, for example, problems involving fluid flow through porous media (Cottle [7]), journal bearing lubrication problems (Cottle [7], Crank [9]), elastic-plastic torsion problems and maximizing oil production problems (Bershchanskii and Meerov [4]). We note that solutions for the above problems obtained by the approach of the theory of complementarity problem are in fact approximate solutions using finite difference method.

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A new problem called the *generalized quasi-variational inequality problem* is introduced. This new formulation extends all kinds of variational inequality problem formulations that have been introduced and enlarges the class of problems that can be approached by the variational inequality problem formulation. Existence results without convexity assumptions are established and topological properties of the solution set are investigated. A new problem call the *generalized implicit complementarity problem* is also introduced which generalizes all the complementarity problem formulations that have been introduced. Applications of generalized quasi-variational inequality and implicit complementarity problems are given.